Problem II.1

A quantum particle of mass m is subject to an attractive three-dimensional central potential

$$V(r) = -\left(\frac{\hbar^2}{2m}\right)V_0\alpha e^{-\alpha r^2}.$$

Notice that, here, V_0 is unitless.

(a) [**7 points**] By examining the radial Schrödinger equation and using clear arguments based on the form of the attractive potential, show that for fixed $V_0 > 0$ and α there are values of the angular momentum quantum number ℓ such that no bound states exist for that ℓ. Obtain a formula for the range of ℓ values (in terms of V_0, α, m) for which there *definitely* are no bound states.

(b) [**7 points**] Using the integral form of the Schrödinger equation for the ground state wave function, ψ_0,

$$\psi_0(r) = \frac{m^2}{2\pi\hbar^2}\int\frac{e^{-k|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}(-V(r'))\psi_0(r')d^3r', \tag{1}$$

where $k = \sqrt{2m(-E)}/\hbar$, prove that for a *general* spherically symmetric potential $V(r)$, obeying the bounds $V(r) \le 0$ and $V(r) \to 0$ as $r \to \infty$, the following inequality is a *necessary* condition for the existence of bound states:

$$\int_0^\infty r|V(r)|dr \ge \frac{\hbar^2}{2m}$$

(c) [**4 points**] Using the inequality given in part (b) above (just accept it as correct even if you could not derive it), obtain the necessary condition for the existence of bound states in the attractive potential used in part (a).

(d) [**7 points**] Consider now the *one-dimensional* Gaussian attractive potential

$$V(x) = -\left(\frac{\hbar^2}{2m}\right)V_0\alpha e^{-\alpha x^2}.$$

Using the variational wavefunction $\psi(x) = Ae^{-\beta x^2/2}$, find a cubic algebraic equation for β that leads to an upper bound for the ground state energy.
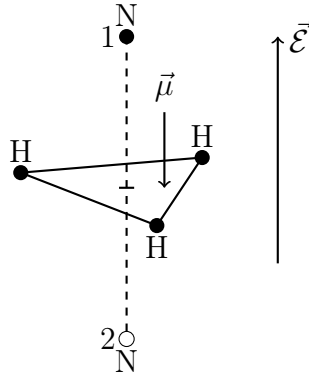
Hint:

$$\int_0^\infty y^n e^{-b^2y^2}dy = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2b^{n+1}}$$

where $\Gamma(n) = (n-1)!$ and $\Gamma(n+\frac{1}{2}) = \{(1)\cdot(3)\cdot(5)\cdot...\cdot(2n-1)\}\frac{\sqrt{\pi}}{2^n}$ for integer n.

Problem II.2

Consider the ammonia (NH_3) molecule in its ground state. The three hydrogen atoms are located at the corners of an equilateral triangle, and the nitrogen atom lies either "above" or "below" the H_3-plane on the perpendicular axis of symmetry (see Figure below).

These two configurations define two position states for the nitrogen atom, $|1\rangle$ and $|2\rangle$, and have an associated dipole moment μ which interacts with an external electric field $\mathcal{E}$ as shown. We assume that these two states form a complete set, but they are not eigenstates of the system since there is a coupling causing tunneling back and forth.

(a) [**3 points**] In the absence of tunneling, the position states would have equivalent energies E_0 when the electric field $\mathcal{E} = 0$. Using this information, construct the 2×2 Hamiltonian matrix in the position state basis, including tunneling at a frequency $\Delta/\hbar$.

(b) [**3 points**] Determine the eigenstates for this system and their eigenvalues. Label these states as $|+\rangle$ and $|-\rangle$, and the corresponding energies as E_+ and E_-, respectively, with $E_+ > E_-$.

(c) [**4 points**] If the system is known to be entirely in state $|1\rangle$ at $t = 0$, find the probability for it to be in state $|2\rangle$ as a function of time t.

The mean positions of positive and negative charges are relatively displaced so that the ammonia molecule has an electric dipole moment. The dipole moment has a magnitude μ, and points from N towards the H_3-plane along the symmetry axis. (The Figure shows $\vec{\mu}$ when N is in state $|1\rangle$) .

(d) [**7 points**] Apply an oscillatory electric field $\mathcal{E} = Ae^{i\omega t}$, along the symmetry axis pointing towards position 1. Assuming the electric field does not change the geometry of the molecule, write down the corresponding perturbation Hamiltonian H' in the $(|1\rangle, |2\rangle)$ basis, and convert it to the $(|+\rangle, |-\rangle)$ basis.

(e) [**8 points**] Define the time-dependent probability amplitudes $C_+(t)$ and $C_-(t)$ for finding the system in the states $|+\rangle$ and $|-\rangle$, respectively, by the expansion of the state vector

$$|\Psi(t)\rangle = C_+(t)|+\rangle + C_-(t)|-\rangle. \tag{1}$$

Derive the exact, decoupled equations governing the time-variation of $C_+(t)$ and $C_-(t)$.

Problem II.3

Consider scattering of a particle of mass m and momentum k on a spherically symmetric attractive potential $V = V_0 < 0$ for $r < R$ and $V = 0$ for $r > R$.

(a) [**5 points**] Calculate the scattering amplitude in the Born approximation.

(b) [**5 points**] Find the low energy limit of this result.
   Using the ratio of the scattering wave-function amplitude to the incoming one, determine under what condition (on V_0, R) the Born approximation is valid in the low-energy limit.

(c) [**5 points**] Now consider s-wave scattering without approximation. Obtain an equation for the s-wave scattering phase shift if $V_0 < 0$.

(d) [**5 points**] Solve for the scattering length a as a function of V_0 in the low-energy limit.
   At what minimum value of $|V_0|$ does the scattering length approach infinity?
   What is the scattering phase shift at this V_0?

(e) [**5 points**] For this attractive potential, do you think there can be a scattering resonance for s-wave scattering? What about p-wave scattering?

Problem II.4

Two identical, electrically neutral, spin-1 bosons of mass M interact solely via a spin-dependent potential of the form $V = a\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{r}$ where a is a positive constant and r is the distance between the two particles; $\mathbf{S}_1$, $\mathbf{S}_2$ are spin operators for the two particles. You should ignore center of mass motion throughout the problem.

The Hamiltonian commutes with the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, the orbital angular momentm $\mathbf{L}$ and the total angular momentum $\mathbf{J} = \mathbf{S} + \mathbf{L}$. Thus the eigenstates of the Hamiltonian may be labeled by $|n, s, l, j, m\rangle$, where $n \geq 1$ is an integer specifying the principal quantum number and s, l, j, m have their usual meaning. In this problem your principal task is to determine the allowable values of n, s, l, j, m and the energies associated with them.

(a) [**5 points**] The interaction depends on $\mathbf{S}_1 \cdot \mathbf{S}_2$. As a first step, find the the eigenvalues of this operator in terms of the quantum number s that specifies the total spin.

(b) [**5 points**] Show that bound states have $s = 0$ or $s = 1$ but not $s = 2$.

(c) [**5 points**] For $s = 0$, find all allowable n, l, j and m quantum numbers and the energy of the system for these levels.

(d) [**5 points**] For $s = 1$, find all allowable n, l, j and m quantum numbers and the energy of the system for these levels.

(e) [**5 points**] Determine the degeneracy of all levels with $n \leq 3$ for both $s = 0$ and $s = 1$.

Potentially useful facts:

- The energy levels of a charged scalar with mass M_s in a central Coulomb potential, $V = -\frac{q^2}{r}$, has bound states $|n, l, m\rangle$ with $n = 1, 2, 3, ...$ $l = 0, 1, ..., n-1$ and $m = -l, -l+1, ..., l-1, l$, with energies only depending on n and given by $E_n = -\frac{M_s q^4}{2\hbar^2 n^2}$.

- $\langle j_1 j_2 m_1 m_2 | sm \rangle = -1^{j_1 + j_2 - J} \langle j_1 j_2 m_2 m_1 | sm \rangle$ where $\langle j_1 j_2 m_1 m_2 | sm \rangle$ is a Clebsch-Gordan coeffcient.

Problem II.5

The Debye model of lattice vibrations postulates an excitation spectrum of the form $E(k) = \hbar k \nu = \hbar \omega$ for $\omega \leq \omega_D$, where ω_D is the Debye frequency and ν is the speed of sound. This model applies to materials of any dimension, where we can assume for simplicity that ν is the same in all directions.

(a) [**7 points**] Calculate the density of states $\rho(\omega)$ and resultant temperature dependence of heat capacity $C(T)$ for three-dimensional vibrations in a monatomic solid in the limit $T \ll \Theta_D$, the Debye temperature. [Hint: you do not have to evaluate integrals leading to numerical coefficients for the heat capacity.]

(b) [**7 points**] Do the same for two-dimensional vibrations.

(c) [**7 points**] Do the same for one-dimensional vibrations.

(d) [**4 points**] Explain in detail how you would experimentally measure the heat capacity of a material, including practical considerations that would be required in a real experiment.