UNIVERSITY OF MARYLAND

Department of Physics

College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION

PART A

August 23, 2021	**10:00 am – 12:00 pm**
August 24, 2021	**10:00 am – 12:00 pm**

Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet – not your name!

You may keep this packet with the questions after the exam.

Problem A.1

A surprisingly accurate approximation for the motion of a mass m orbiting a black hole of mass M can be obtained by using ordinary nonrelativistic Newtonian mechanics, but slightly modifying the usual $1/r$ potential to

$$U(r) = -\frac{GmM}{(r - r_g)}. \tag{1}$$

Here G is the gravitational constant, and r_g is the radius of the black-hole event horizon. Orbits with $r < r_g$ are inside the black hole and, so, unphysical. Assume that $M \gg m$, so the black hole is stationary.

(a) **[6 points]** By means of a Lagrangian or otherwise, obtain, for a general potential $U(r)$ and orbital angular momentum ℓ, an equation for the radius $r(t)$ in the form

$$m\frac{d^2r}{dt^2} + \frac{\partial}{\partial r}W_{\text{eff}}(r, \ell) = 0, \tag{2}$$

with an explicit formula for the effective radial potential $W_{\text{eff}}(r, \ell)$.

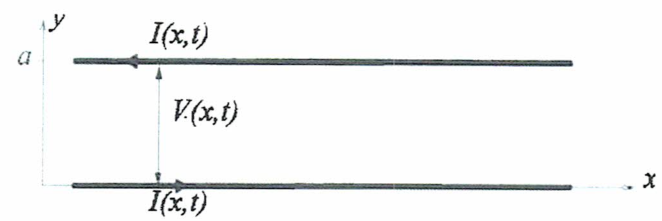
(b) **[6 points]** Using the potential $W_{\text{eff}}(r, \ell)$ obtained above, find the value of ℓ that allows a circular orbit of a given radius r_0 around the black hole.

(c) **[6 points]** Explain how you would use some property of $W_{\text{eff}}(r, \ell)$ to determine whether the circular orbit found in Part (b) is stable or unstable, when the particle is given a small kick that does not alter its orbital angular momentum ℓ.

(d) **[7 points]** On the basis of Part (c), find the critical radius r_c separating stable circular orbits for $r_0 > r_c$ from unstable orbits in the range $r_g < r_0 < r_c$ for the potential $U(r)$.

Problem A.2

Consider a very long transmission line of length ℓ consisting of two perfectly conducting wires running parallel to the x axis at $y = 0$ and $y = a > 0$ in vacuum, as shown in the figure. The wires have capacitance $\tilde{C} = C/\ell$ and inductance $\tilde{L} = L/\ell$ per unit length. This problem concerns electromagnetic waves in this system with wavelengths λ much longer than the distance a between the wires: $\lambda \gg a$, but much shorter than the wires length: $\lambda \ll \ell$.



(a) **[5 points]** Let $V(x,t)$ be voltage between the wires, defined as the integral of the electric field along a straight path between the wires at fixed x and t

$$V(x,t) = -\int_0^a E_y(x,y,t)\,dy. \tag{1}$$

Show that

$$\frac{\partial V(x,t)}{\partial x} = \kappa \frac{\partial I(x,t)}{\partial t} \tag{2}$$

with some constant κ depending on $\tilde{L}$ and/or $\tilde{C}$ that you should find. Here $I(x,t)$ is the current at x in the lower ($y = 0$) wire, whereas the current in the upper ($y = a$) wire has the same magnitude, but opposite direction.

Hint: The magnetic flux through a rectangular area bounded by the wires and a very small horizontal width Δx is $\Delta\Phi_B = I(x,t)\,\tilde{L}\,\Delta x$. Apply Faraday's law to this area, taking into account that the electric field is zero inside an ideal wire of zero resistance.

(b) **[5 points]** By using the charge continuity equation, derive a similar relation between $\partial V(x,t)/\partial t$ and $\partial I(x,t)/\partial x$.

Hint: The electric charges induced in short segments of width Δx at the top and bottom wires are related to capacitance as $\Delta Q = \pm V(x,t)\,\tilde{C}\,\Delta x$.

(c) **[5 points]** Combining the differential equations from Parts (a) and (b), obtain a wave equation for $I(x,t)$ and express the wave velocity v in terms of $\tilde{L}$ and $\tilde{C}$.

(d) **[5 points]** For the wave $I(x,t) = I_0 \sin[k(x - vt)]$ propagating to the right, find the corresponding $V(x,t)$.

Suppose $I_0 > 0$, and $\sin[k(x - vt)] = 1$ for given values of x and t. According to Eq. (1), what is the direction of $E_y(x,y,t)$ for the same x and t: up or down in the figure?

(e) **[5 points]** Suppose the right end of the transmission line is terminated by a resistor R connecting the two wires at $x = \ell$. Find the value of R such that the wave $I(x,t) = I_0 \sin[k(x - vt)]$ propagating to the right is completely absorbed at $x = \ell$ and does not generate a reflected wave.

Problem A.3

Consider a particle of mass m in an asymmetric one-dimensional potential of width a and depth $V_0 > 0$:

$$V(x) = \begin{cases} \infty, & -\infty < x < 0, \\ -V_0, & 0 < x < a, \\ 0, & a < x < \infty. \end{cases} \tag{1}$$

(a) [**9 points**] Derive a transcendental equation that determines the energies E of bound states for this potential.

(b) [**4 points**] What is the minimum depth V_0 for which a bound state exists?

(c) [**4 points**] How many bound states are there for a general depth V_0?

(d) [**4 points**] Suppose the system has a shallow bound state with the energy $E = -0.01\,\hbar^2/2ma^2$. Estimate the probability P to find the particle inside the potential well, with the coordinate $0 < x < a$.

(e) [**4 points**] Suppose the potential (1) has bound states. Now let us modify the potential by adding a positive part outside of the negative part:

$$V(x) = \begin{cases} \infty, & -\infty < x < 0, \\ -V_0, & 0 < x < a, \\ +W_0, & a < x < b, \\ 0, & b < x < \infty, \end{cases} \tag{2}$$

where $W_0 > 0$.

Describe qualitatively how the presence of the positive part modifies energies of the bound states compared with the case $W_0 = 0$. Are they lowered, raised, or unchanged?

Problem A.4

This problem deals with the first quantum mechanical model that can account for the observation that the heat capacity per volume $C_V(T)$ of insulators is much smaller at low temperatures than the classical result (the Dulong-Petit Law).

(a) [**4 points**] Derive an expression for the average **energy** at temperature T of a **quantum** harmonic oscillator of the angular frequency ω in **one dimension** ($D = 1$).

(b) [**4 points**] Mindful of Planck's results and the quantum theory of oscillators, Albert Einstein proposed a crude model of insulators. He set all the vibrational modes of the N atoms in a three-dimensional ($D = 3$) solid insulator of volume V to have the same frequency ω_E. Find $C_V(T)$ of this so-called Einstein model.

(c) [**2 points**] How would $C_V(T)$ change if the problem were formulated in one- or two-dimensional space?

(d) [**3 points**] Find the high-temperature limit of C_V in $D = 3$ and verify that it agrees with the classical result coming from the Equipartition Theorem.

(e) [**3 points**] Find the expression to which $C_V(T)$ simplifies at temperatures well below $\hbar\omega_E/k_B$, and then evaluate $C_V(0)$.

(f) [**2 points**] Sketch the behavior of $C_V(T)$ vs. T from $T = 0$ to a temperature a few times $\hbar\omega_E/k_B$.

(g) [**2 points**] Phonon modes in a solid can be acoustic or optical. Which of these modes are better described by the Einstein model?

(h) [**3 points**] Why does the Einstein model poorly describe the magnitude and thermal behavior of $C_V(T)$ of a metal?

(i) [**2 points**] Within the Einstein model, compare two single-element insulators, both having the same N and V but with each made entirely of one of two different isotopes of the element. How, if at all, would $C_V(T)$ differ? In the limits of high and of low temperature, would C_V be larger or smaller for the insulator with the higher-mass isotope?