UNIVERSITY OF MARYLAND

Department of Physics

College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION

PART A

August 17, 2022	**10:00 am – 12:00 pm**
August 18, 2022	**10:00 am – 12:00 pm**

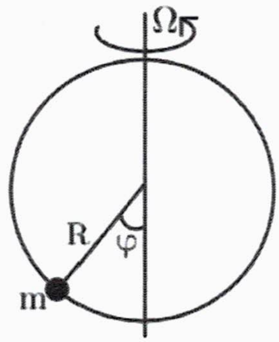
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet – not your name!

You may keep this packet with the questions after the exam.

Problem A.1

A particle of mass m is constrained to move on a circular wire of radius R, which spins with constant angular velocity Ω about a vertical diameter. The particle can slide without friction along the wire and feels the vertical force of gravity mg. The position of the particle is characterized by the angle φ as shown in the Figure.



(a) **[3 points]** Find the Lagrangian for the system using φ as the generalized coordinate and obtain the corresponding Euler-Lagrange equation.

(b) **[4 points]** (i) Construct the Hamiltonian corresponding to this system and write it in terms of the generalized momentum and coordinate.

(ii) Identify the effective potential of the system and sketch it as a function of φ for two different nonzero values of Ω where it has qualitatively different forms.

(c) **[3 points]** (i) Is the Hamiltonian a constant of motion?

(ii) How is the Hamiltonian related to the energy of the system? Is the energy conserved?

(d) **[7 points]** (i) Identify the equilibrium points φ_* and discuss their stability.

(ii) Determine how the stable equilibrium points depend on Ω, and find the critical value Ω_c for which a qualitative change occurs. What happens in the limit $\Omega \to \infty$?

(e) **[8 points]** (i) Find the frequency of small oscillations around the stable equilibria for $|\Omega| < \Omega_c$ and $|\Omega| > \Omega_c$.

(ii) What happens to the oscillation frequency when $|\Omega| \to \Omega_c$?

Problem A.2

This problem concerns a quantum particle of mass m in one dimension ($-\infty < x < \infty$) subject to a linear potential $V(x) = \lambda x$ with $\lambda > 0$. The energy eigenstate wave function $\psi_E(x)$ will be found approximately by transforming from the exact momentum-space wave function, using the saddle point method. It suffices to find the $E = 0$ eigenfunction, since those for other values of E are simply related by a shift of the x coordinate.

(a) **[4 points]** (i) Write down the time-independent Schrödinger equation for an energy eigenvalue E.

(ii) Graph the potential $V(x)$ and make a rough sketch of the $E = 0$ energy eigenfunction $\psi_0(x)$ on the same graph, showing where the wave function is oscillating and where it is not, and indicating any position dependence of the oscillation wavelength.

(b) **[3 points]** (i) Express the time-independent Schrödinger equation in the momentum representation, and solve it for the $E = 0$ energy eigenfunction $\phi_0(p)$. (There is no need to normalize the wave function in this problem.)

(ii) Using your solution $\phi_0(p)$, write down the corresponding wave function $\psi_0(x)$ in the position representation, as an integral over p.

Hint: The position operator is $\hat{x} = i\hbar\frac{d}{dp}$ in the momentum representation.

The p-integral for the zero-energy eigenfunction is

$$\psi_0(x) = \int_{-\infty}^{\infty} dp\, e^{ig(p)}, \qquad g(p) = px + \tfrac{1}{3}p^3,$$

in units with $\hbar = 2m = \lambda = 1$ (which you may adopt). This integral is proportional to the Airy function, but for the purposes of this problem no prior knowledge of the Airy function is needed. For sufficiently large values of $|x|$, the integral can be evaluated accurately using the saddle-point method.

(c) **[9 points]** Show that for $x > 0$ there are two purely imaginary saddle point values $p = p_*$, and write those values.

Find an approximate form of $\psi_0(x)$ for $x > 0$ by deforming the contour of p integration to pass, along the direction of steepest descent, through the saddle point that yields a good approximation to the integral via a truncated Taylor expansion about the saddle point. (*Hint*: This is the saddle point at which the value of the integrand does *not* diverge at $x \to +\infty$.)

(d) **[9 points]** Show that for $x < 0$ there are two real saddle point values $p = p_*$, and write those values.

Find an approximate form of $\psi_0(x)$ for $x < 0$ by deforming the contour to pass, along directions of steepest descent, through *both* saddle points, and adding the two corresponding approximate contributions to the integral.

Possibly useful formula:

$$\int_{-\infty}^{+\infty} e^{-au^2}\, du = \sqrt{\frac{\pi}{a}}.$$

Problem A.3

In this problem you don't need to do any calculus or solve differential equations. Just use simple algebra based on the known results for a one-dimensional (1D) harmonic oscillator.

(a) **[5 points]** Consider a spinless particle of mass μ and electric charge e subject to a two-dimensional (2D) harmonic-oscillator potential with the Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2\mu} + \frac{1}{2}\mu\omega_0^2\left(x^2 + y^2\right), \tag{1}$$

where ω_0 the oscillator frequency.

Treating $H = H_x + H_y$ as the sum of two 1D oscillators, obtain the energy spectrum E_n, where the integer $n = 0, 1, 2, \ldots$ labels successive energy levels, and find the degeneracy for each n.

(b) **[5 points]** The Hamiltonian (1) has rotational symmetry about the z axis perpendicular to the xy plane, so it commutes with the angular momentum operator L_z, whose eigenvalues are $m\hbar$ with an integer m. Thus we can find a set of simultaneous eigenstates $|n, m\rangle$ of H and L_z.

(i) What is the maximal $|m|$ for a given n?
(ii) What is the relation between the parities of m and n?
(iii) By matching the degeneracies from Part (a), find the set of m for a given n.

Hints: (i) Writing $x = r\cos\phi$ and $y = r\sin\phi$ in terms of the radius and angle, the oscillator wavefunctions $\psi_{n_x}(x)$ and $\psi_{n_y}(y)$ are polynomials in $\cos\phi$ and $\sin\phi$ of the orders n_x and n_y. On the other hand, the angular dependence can be represented as a superposition of $\psi_m(\phi) = e^{im\phi}$. Find the maximal $|m|$ by matching these two pictures.

(ii) Consider the spacial parity operation, which is represented by either $x \to -x$ and $y \to -y$, or by $\phi \to \phi + \pi$.

(c) **[5 points]** Now suppose a magnetic field B is applied along the z axis. Introduce the vector potential $\boldsymbol{A}$ in the gauge $A_x = -By/2$ and $A_y = Bx/2$ into the Hamiltonian (1). Show that the Hamiltonian describes a 2D oscillator of a modified frequency with an additional term involving the angular momentum operator $L_z = xp_y - yp_x$.

(d) **[5 points]** Argue that, in the presence of the magnetic field B, the integers m and n labeling the eigenvalues of L_z and of the new oscillator Hamiltonian in Part (c), respectively, are still good quantum numbers, and obtain the energy levels $E_{n,m}$.

How do the degenerate energy levels found Part (a) split to the 1st order in a small B?

(e) **[5 points]** Now set $\omega_0 = 0$ while $B \neq 0$ in the Hamiltonian found in Part (c). In this case, show that the energy spectrum $E_{n,m}$ obtained in Part (d) reproduces the Landau levels $E_N = (N + 1/2)\,\hbar\omega_c$. Define ω_c and give its physical interpretation for the corresponding classical system.

How is the integer index $N = 0, 1, 2, \ldots$ of the Landau levels related to n and m? What is the degeneracy for each N? What is the minimal value of n for a given N?

Problem A.4

When an ensemble of identical bosons is brought to a sufficiently low temperature T and high density n, there can be a macroscopic occupation of the ground state known as Bose-Einstein condensation. We shall consider this effect for cold atoms in a three-dimensional *harmonic trap* with the potential $\frac{1}{2}m\omega^2 r^2$, where m is the single particle mass, r is the distance from the center of the trap, and ω is the oscillator frequency. Assume that the bosons have no internal degrees of freedom, and that mutual interaction can be neglected, and take $k_B T \gg \hbar\omega$ so that the density of states can be treated as continuous.

(a) **[5 points]** The single-particle density of states is defined as $D(\varepsilon) = d\mathcal{N}/d\varepsilon$ where $\mathcal{N}(\varepsilon)$ is the number of states with energy less than ε. Show that for the harmonic trap

$$D(\varepsilon) = \frac{\varepsilon^2}{2(\hbar\omega)^3}. \tag{1}$$

Hint: The number of distinct states corresponds to the phase-space volume in units of $(2\pi\hbar)^3$. The phase-space region of states below a given energy is a six-dimensional ball when expressed in suitable units, and the volume of a 6-ball of radius R is $\pi^3 R^6/6$.

(b) **[5 points]** Derive a relationship between the critical temperature of Bose-Einstein condensation T_c and the number of particles N.

Hint: Recall that Bose condensation is achieved as the chemical potential $\mu \to 0$, when the minimal energy available to the atoms is $\varepsilon = 0$. Enforce the constraint of a fixed N on the Bose gas in the limit $\mu \to 0$.

(c) **[5 points]** Calculate the temperature dependences of the energy $U(T)$ and the number $N_*(T)$ of the uncondensed atoms for $T \leq T_c$.

What are the limiting values of U and N_* at $T = 0$ and $T = T_c$?

(d) **[5 points]** Calculate the single-particle mean value $\langle r^2 \rangle$ at the critical temperature T_c.

Hint: Use $U(T_c)/N$ from Part (c) and the fact that the mean kinetic and potential energies are equal for a particle in a harmonic potential.

(e) **[3 points]** From Part (d), estimate the spatial volume V occupied by the particles at T_c.

Then find how T_c depends on the number density $n = N/V$.

(f) **[2 points]** Determine whether Bose-Einstein condensation would happen in a harmonic potential in two-dimensional and in one-dimensional space.

Possibly useful information:

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = n!\,\zeta(n+1), \qquad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \qquad \zeta(1,\,2,\,3,\,4,\,5) \approx \{\infty,\,1.64,\,1.20,\,1.08,\,1.04\}$$

$\zeta(s)$ is called the Riemann zeta-function.