UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION PART II

September 3, 2020 11:00 a.m. – 3:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II.1

Consider the problem of a heavy quark-antiquark bound state. Let the Hamiltonian be written, in the CM system, in a nonrelativistic form

$$H = \frac{\vec{p}^2}{2\mu} + V(\vec{r}), \tag{1}$$

where $\vec{r} = \vec{r}_{quark} - \vec{r}_{antiquark}$ and μ is the reduced mass of the quark and antiquark pair.

(a) [**10 points**] Derive a relationship between the average kinetic energy and the average potential energy for the bound state. (Hint: Evaluate $\frac{d}{dt}\langle \vec{r} \cdot \vec{p} \rangle$).

(b) [**5 points**] The Feynman-Hellmann Theorem states that if the Hamiltonian H depends on a parameter λ, $H = H(\lambda)$, and if $|\psi(\lambda)\rangle$ is an eigenstate of $H(\lambda)$,

$$H(\lambda)|\psi(\lambda)\rangle = E(\lambda)|\psi(\lambda)\rangle, \tag{2}$$

then

$$\frac{\partial}{\partial \lambda} E(\lambda) = \left\langle \psi(\lambda) \middle| \left(\frac{\partial}{\partial \lambda} H(\lambda) \right) \middle| \psi(\lambda) \right\rangle \tag{3}$$

Prove the theorem.

(c) [**6 points**] Suppose

$$V(r) = V_0 \ln(r/r_0) \tag{4}$$

Deduce the dependence of the bound state energies E_n on the reduced mass μ.

(d) [**1 points**] From your results in (c) what, if anything, can you say about the dependence of $E_n - E_m$ on the reduced mass μ. (Here n and m refer generically to the quantum numbers of the bound states.)

(e) [**3 points**] Estimate the radius of the ground state wavefunction, up to an unspecified factor of order unity.

Problem II.2

Consider a quantum particle of mass m in a potential

$$V(x) = \frac{m\omega^2}{2}x^2 + cx^3. \tag{1}$$

We first consider the case with $c = 0$, and then add the x^3 term using perturbation theory.

(a) [**4 points**] What are the energy levels for the particle in the potential $V(x)$ when $c = 0$?

(b) (i) [**4 points**] For what values of n and n' is the matrix element $\langle n'|x^3|n\rangle$ between the unperturbed energy eigenstates nonzero? ($n = 0$ labels the ground state.)

(ii) [**4 points**] Write explicit expressions for the nonzero matrix elements $\langle n'|x^3|n\rangle$.

(c) [**4 points**] What is the correction to the energy levels of Part (a) in first-order perturbation theory, where cx^3 is the perturbation?

(d) [**5 points**] Find the second-order correction to the ground state energy.

(e) [**2 points**] For the potential $V(x)$ of (1), the particle actually has no ground state. Explain why not, and describe the conditions under which it nevertheless makes physical sense to use perturbation theory for the ground state.

(f) [**2 points**] What condition on c should be satisfied if second-order perturbation theory is to give a good approximation for the ground state?

You might find helpful the equation

$$\langle n'|x|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$$

where $|n\rangle$ and $|n'\rangle$ are orthogonal eigenstates of the unperturbed Hamiltonian.

Problem II.3

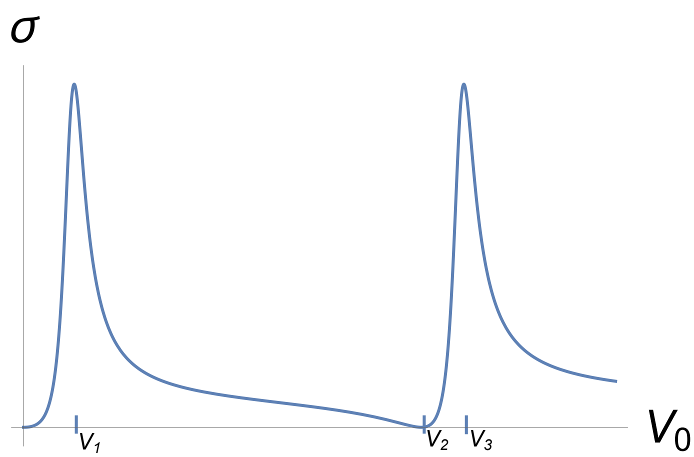
A particle of mass m and energy $E = \hbar^2 k^2/2m$ scatters from a spherically symmetric potential,

$$V(r) = -V_0 \quad \text{for} \quad r < R, \qquad V(r) = 0 \quad \text{for} \quad r > R. \tag{1}$$

In this problem we assume that the particle has very low energy E, such that $kR \ll 1$.

(a) [**8 points**] Assuming that the strength V_0 is not sufficient to support a bound state, find the cross section σ in the limit $E \to 0$, as a function of the remaining parameters, V_0, R, m, and $\hbar$.

(b) [**4 points**] As the depth of the potential, V_0, is increased while keeping the energy E fixed, the total cross section behaves as in the Figure below.



Find the value of the total cross section at the maxima, where $V_0 = V_1$ and $V_0 = V_3$, in terms of m, E, and $\hbar$.

(c) [**7 points**] Determine approximately the values V_1 and V_3 at which the first two maxima occur. *Hint*: Consider the values of V_1 and V_3 in the limit of zero energy.

(d) [**6 points**] What is the approximate value of V_2 at which the cross section is zero? The solution involves a transcendental equation. Sketch and label a graph showing the location of the relevant root, and give explicit upper and lower bounds for V_2.

Possibly useful formula connecting the amplitude f_l and the phase δ_l in the scattering channel with the orbital angular momentum l

$$f_l = \frac{e^{i\delta_l} \sin \delta_l}{k}.$$

Problem II-4

Two particles interact via a spin-spin Hamiltonian $A\mathbf{S}_1 \cdot \mathbf{S}_2$ where A is a positive constant and $\mathbf{S}_{1,2}$ are the spin angular momenta of the two particles. Particle 1 has spin 1 and a magnetic moment of $\mu_1 = -\frac{\mu_B}{\hbar}\mathbf{S}_1$, whereas Particle 2 has spin $\frac{1}{2}$ and magnetic moment zero.

(a) [**6 points**] What are the energy levels of the system and the degree of degeneracy of the levels? Give a detailed derivation of the possible results.

(b) [**8 points**] Write a basis of normalized energy eigenstates corresponding to the different energy levels in part (a) as linear combinations of products of single-particle spin states.

(c) [**7 points**] If the system is placed in a magnetic field of strength B aligned with the z-axis, what then are the approximate energy eigenstates and eigenvalues if $B \gg A\hbar^2/\mu_B$? Use the product of single particle spin states as before.

(d) [**4 points**] Are any of the levels exactly linear in B for *all* $B > 0$? If not, explain why not. If so, which levels are these and what are their energies as a function of B ?

Possibly useful formula: $J_\pm|j,m\rangle = \sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle$.

Problem II.5

Consider a free-electron gas, with N electrons and dispersion relation $\epsilon = (\hbar|\mathbf{k}|)^2/2m$ at temperature T=0, in $d = 3$ or $d = 2$ dimensions, contained in a volume $V = L^3$ or area $A = L^2$, respectively, so a cube or a square.

(a) [**6 points**] The number of single-particle energy eigenstates (counting all degeneracies) between ϵ and $\epsilon + d\epsilon$ is $G(\epsilon)\, d\epsilon$, where $G(\epsilon)$ is known as the density of states. Show that $G(\epsilon)$ satisfies

$$G(\epsilon) \propto \epsilon^{\alpha}\,, \tag{1}$$

finding the value of α for $d = 3$ and for $d = 2$. Assume throughout that L is sufficiently large that finite-size effects can be ignored. Ignore numerical factors and dimensionful constants, since you seek only the ϵ dependence of $G(\epsilon)$. (*Hint*: With periodic boundary conditions each single-particle state can be taken to have a definite wave vector.)

(b) [**8 points**] i) (1) What are the units of $G(\epsilon)$?

(2) Explain why the proportionality relation (1) can be written more specifically as the following equation:

$$G(\epsilon) = \frac{BN\epsilon^{\alpha}}{{\epsilon_F}^{\alpha'}}, \tag{2}$$

where ϵ_F is the Fermi energy and B is a numerical constant. Specify how the numerical value of α' is related to that of α. (*Note*: To answer this question one does not need to determine ϵ_F in terms of the parameters of the problem.)

ii) Find the value of B for $d = 3$ and for $d = 2$.

iii) For N electrons, how does ϵ_F depend on N and V, and on N and A for $d = 2$? [Again, prefactors are not needed, just the proportionality with the correct exponents.]

For graphene (a single planar sheet of graphite) near the Dirac points, the electronic dispersion relation can be written $\epsilon = \hbar v_s|\mathbf{k}|$ (where $\mathbf{k}$ is measured from a Dirac point, a fact you can ignore here).

(c) [**6 points**] i) Show that the density of states $G(\epsilon)$ is still proportional to $\epsilon^{\alpha}/{\epsilon_F}^{\alpha'}$, and find the new value of α for $d = 2$.

ii) Does the relationship of α' to α change from part 2? If yes, how? If not, why not?

(d) [**5 points**] The total low-temperature heat capacity of a metal with fixed volume V [or area A] and fixed N is known to behave as a power-law of T. Give a quick argument to show what this power is. It may be helpful to sketch the change in the Fermi-Dirac distribution when T increases slightly from 0. (Use of the Sommerfeld expansion is not intended!)