UNIVERSITY OF MARYLAND

Department of Physics

College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION

PART B

January 19, 2022	**10:00 am – 12:00 pm**
January 20, 2022	**10:00 am – 12:00 pm**

Problem B.1

Consider an electrically-neutral (x, y) plane that carries a time-dependent but spatially-uniform surface current density $\boldsymbol{K}(t) = \hat{\boldsymbol{x}}\, K_0 \cos(\omega t)$ flowing along the x direction. The oscillating current induces two plane electromagnetic waves propagating away from the current sheet, along $+z$ for $z > 0$ and along $-z$ for $z < 0$. Assume boundary conditions such that the electromagnetic field is composed solely of these waves.

(a) [**5 points**] Find the magnetic fields $\boldsymbol{B}_+(t)$ and $\boldsymbol{B}_-(t)$ at the distances $z = 0^+$ and $z = 0^-$ on the opposite sides of the current sheet in the immediate vicinity of it.

Hint: Use the symmetry and boundary conditions for a current sheet.

(b) [**5 points**] Obtain the fields $\boldsymbol{B}_\pm(z, t)$ and $\boldsymbol{E}_\pm(z, t)$ of the electromagnetic waves for $z > 0$ and $z < 0$.

Make sure they match the boundary conditions $\boldsymbol{B}_\pm(t)$ at $z = 0^\pm$ from Part (a). What is the boundary condition for $\boldsymbol{E}_\pm(z, t)$ at $z = 0$, assuming that the current sheet in infinitesimally thin and electrically neutral?

(c) [**5 points**] Find the vector potentials $\boldsymbol{A}_\pm(z, t)$ describing the two propagating waves for $z > 0$ and $z < 0$ in the gauge where $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ and the scalar potential is zero.

(d) [**5 points**] Obtain the Pointing vectors $\boldsymbol{S}_\pm(z, t)$ and the (time-averaged) intensities $I_\pm$ of the two plane waves.

(e) [**5 points**] Write a general expression for the work done per unit time per unit area by an electric field on a surface current density, and calculate the time average of this quantity for the surface current in this problem. Is it positive or negative?

Compare the result with the energy intensity carried away by the electromagnetic waves from Part (d) and give a physical interpretation.

Problem B.2

This problem looks at some collision processes that can change the energy of an astrophysical photon.

(a) [**4 points**] Consider two photons of energies E_1 and E_2 colliding at an angle θ.

Derive a threshold condition (on these energies) necessary to produce an electron-positron pair.

For fixed E_2, what is the minimal threshold for E_1, and at what angle θ is it achieved?

(b) [**2 points**] Estimate the minimal energy at which a photon traveling through space can produce an electron-positron pair upon collision with a photon of energy 6×10^{-4} eV from the Cosmic Background Radiation. (The electron mass is 0.5 MeV.)

(c) [**10 points**] Now consider scattering of a photon and an electron. Assume that the electron and the photon are initially moving towards each other on the same axis.

Find the energy E_f of the scattered photon in terms of the initial photon energy E_i and the angle θ of photon scattering, as well as the speed v and mass m_e of the electron.

Hint: Express the square (i.e. inner product with itself) of the electron 4-momentum after scattering in terms of the other 4-momenta in the problem.

(d) [**2 points**] Taking the limit $v = 0$ in your solution to Part (c), recover the standard Compton relation between the final λ_f and initial λ_i wavelengths of the photon.

(e) [**7 points**] A high-energy electron can transfer a large energy to the photon in a so-called *inverse Compton scattering* process. Obtain E_f in the limit where the photon scattering angle is π and the electron speed v is close to the speed of light c.

Problem B.3

A spinless particle of mass m and charge q is confined to the xy plane, in a two-dimensional harmonic oscillator potential and a uniform magnetic field B in the $\hat{z}$ direction. The Hamiltonian in a suitable gauge takes the form

$$H = \frac{p_x^2 + p_y^2}{2m} - \frac{\omega_B}{2} L_z + \frac{m\omega_B^2}{8}(x^2 + y^2) + \frac{m\omega^2}{2}(x^2 + y^2),$$

where $\omega_B = qB/mc$ is the gyrofrequency and L_z is the z component of angular momentum.

(a) [**3 points**] If $B = 0$, what are the three lowest energy levels and their degeneracies? Give a basis for the states in each of these levels, using the notation $|n_x n_y\rangle$, where n_x and n_y are the excitation numbers for the x and y components of oscillation.

(b) [**5 points**] Find the energy shift of the ground state energy to lowest order in B. Is the system diamagnetic or paramagnetic in this state?

(c) [**5 points**] (i) Find the energies and eigenstates of the first excited level to lowest order in B. (ii) Given that the energy shift of these perturbed eigenstates is linear in the magnetic field, they evidently have a permanent magnetic moment. What is the magnitude of that magnetic moment?

(d) [**5 points**] (i) Show that the approximate energy eigenstates found in part (c) are exact energy eigenstates, and are eigenstates of angular momentum L_z. (ii) Referring to the full spectrum, prove that in fact there exists a complete basis of exact energy eigenstates that are also L_z eigenstates.

(e) [**7 points**] Now consider the anisotropic oscillator, with

$$\omega_x = \omega, \qquad \omega_y = \omega + \Delta.$$

(i) Treating both the magnetic field and the anisotropy as perturbations, find the energy shift(s) of the first excited level at lowest nonvanishing order in B and Δ. (Check that your result has the correct limits when either B or Δ vanishes.) (ii) Sketch a graph of the perturbed first excited energy levels as functions of Δ, with a fixed nonzero magnetic field B. Include both signs of Δ.

Possibly useful formulae:

$$L_z = i\hbar(a_y^\dagger a_x - a_x^\dagger a_y),$$

where the operators are raising and lowering operators for the x and y modes of the oscillator, e.g.

$$a_x = (x/x_0 + ix_0 p_x/\hbar)/\sqrt{2}\,, \qquad x_0 = \sqrt{\hbar/m\omega_x}\,.$$

Problem B.4

An electron of mass m and charge e is subject to a three-dimensional harmonic oscillator potential $m\Omega^2\boldsymbol{r}^2/2$, where $\boldsymbol{r}$ is the electron coordinate. In the notation $|n_x, n_y, n_z\rangle_{\text{el}}$ referring to the oscillator quantum numbers along the x, y, and z axes, the electron is initially in the excited state $|\text{initial}\rangle_{\text{el}} = |0, 0, 1\rangle_{\text{el}}$. By emitting a photon, the electron can make a transition to the ground state. This problem studies the lifetime of the initial state.

The Hamiltonian of interaction between the electron and the vector potential $\boldsymbol{A}$ of an electromagnetic wave is $H_{\text{int}} = -(e/mc)\,\boldsymbol{p}\cdot\boldsymbol{A}$, where $\boldsymbol{p}$ is the electron momentum operator, and the vector potential is the operator

$$\boldsymbol{A} = \sum_{\boldsymbol{k}} \sum_{\lambda=1,2} \sqrt{\frac{2\pi\hbar c}{kV}} \left(a_{\boldsymbol{k},\lambda}\boldsymbol{\varepsilon}_{\boldsymbol{k},\lambda} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + a^{\dagger}_{\boldsymbol{k},\lambda}\boldsymbol{\varepsilon}^{*}_{\boldsymbol{k},\lambda} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \right). \tag{1}$$

Here V is the volume of a large "box" containing the system, and the operators $a^{\dagger}_{\boldsymbol{k},\lambda}$ and $a_{\boldsymbol{k},\lambda}$ create and destroy photons with wave vector $\boldsymbol{k}$ and transverse polarizations $\boldsymbol{\varepsilon}_{\boldsymbol{k},\lambda}$ labeled by $\lambda = 1, 2$.

(a) [**3 points**] Formulate the electric dipole approximation and specify under what conditions it is applicable to this problem.

(b) [**7 points**] Using the Fermi golden rule, obtain the rate of spontaneous emission of a photon with the wave vector $\boldsymbol{k}$ and polarization $\boldsymbol{\varepsilon}_{\boldsymbol{k},\lambda}$.

(c) [**5 points**] Using the result of Part (b) or qualitative arguments, answer these questions:

(i) What is the frequency of the emitted photon?

(ii) Which of the two polarizations $\lambda = 1, 2$ is forbidden for a photon with a given $\boldsymbol{k}$? What is the polarization of the emitted photon?

(iii) How does the emission rate depend on the angle θ between the z axis and $\boldsymbol{k}$? Can a photon be emitted with $\boldsymbol{k}$ parallel to z?

(d) [**7 points**] By integrating the result of Part (b) over all possible wave vectors of the emitted photon, calculate the total transition rate w and the lifetime $\tau = 1/w$ of the excited state.

Verify that your result for τ has the dimensionality of time. (Here we use the Gaussian system of electromagnetic units, where the Coulomb energy is e^2/r without $4\pi\epsilon_0$.)

(e) [**3 points**] Now treat the same oscillator *classically*. According to the Larmor formula of classical electrodynamics, radiated power is $P_c = (2e^2/3c^3)\langle\dot{\boldsymbol{v}}^2\rangle_t$, where $\dot{\boldsymbol{v}}$ is classical acceleration, and the angular brackets indicate time averaging.

Calculate the classical rate of energy loss $w_c = P_c/E_c$ and the lifetime $\tau_c = 1/w_c$, where E_c is classical energy of the oscillator. Compare with the result of Part (d).

Possibly useful information:

The electron momentum operator along z axis, $p_z = i(b_z^{\dagger} - b_z)\sqrt{\hbar\Omega m/2}$, is related to the corresponding raising and lowering operators $b_z^{\dagger}$ and b_z of the oscillator.