UNIVERSITY OF MARYLAND

Department of Physics

College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION

PART B

January 18, 2023 **10:00 am – 12:00 pm**

January 19, 2023 **10:00 am – 12:00 pm**

Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet – not your name!

You may keep this packet with the questions after the exam.

Problem B.1

A dielectric sphere of radius a and dielectric constant ϵ is placed into an otherwise uniform electric field $\boldsymbol{E} = E_0\hat{\boldsymbol{z}}$ applied along the z-axis in vacuum.

(a) [**5 points**] State the equation and boundary conditions needed to determine the electric potential Φ in the presence of the sphere in the static electric field.

(b) [**5 points**] Using spherical coordinates, solve for the potential $\Phi(r, \theta, \phi)$ everywhere inside and outside of the sphere.

(c) [**3 points**] Find the surface charge density $\sigma_p(\theta, \phi)$ on the sphere due to electric polarization.

(d) [**2 points**] Using $\sigma_p(\theta, \phi)$, determine the corresponding electric dipole moment $\boldsymbol{p}$ of the sphere induced by the applied electric field.

(e) [**2 points**] Now consider an electromagnetic plane wave of frequency ω, electrically polarized in the z-direction and propagating in the x-direction, of the form $E_0\hat{\boldsymbol{z}}\cos(kx - \omega t)$. What is the time-averaged Poynting flux S_x^0 of this wave?

(f) [**3 points**] Calculate the time-averaged Poynting flux $S_y(r)$ due to scattering from the dielectric sphere in the y-direction at a distance $r \gg a$ far away from the sphere, in terms of the incoming flux S_x^0 in the plane wave. Assume that the frequency ω is low enough, so that the electrostatic result for the induced dipole moment derived in Part (d) can be used, i.e., the wavelength is much larger than the radius of the sphere.

(g) [**5 points**] The setting Sun appears to be red, but the sky is blue. Why?

Possibly useful information for Part (f):
Far from a z-oriented, harmonically oscillating electric dipole $p\cos(\omega t)$, the electric field in vacuum, expressed in (r, θ, ϕ) spherical coordinates is given in SI units by

$$E_\theta(r, \theta, \phi) = -p\sin\theta\left(\frac{\omega}{c}\right)^2 \frac{\cos[\omega(t - r/c)]}{4\pi\epsilon_0 r}$$

Problem B.2

A plane electromagnetic (EM) wave of frequency ω propagates in vacuum along the positive x direction in the laboratory reference frame denoted by S. The electric field of this wave is

$$\boldsymbol{E}(x,t) = E_0 \hat{\boldsymbol{y}} \cos(kx - \omega t). \tag{1}$$

This incident wave reflects normally from an ideal flat conductor covering the yz plane. The conductor moves in the laboratory frame along the x direction with a velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ comparable with the speed of light c. Express all results in terms of E_0 and $\beta = v/c$.

(a) **[3 points]** How do the coordinates x and t transform from the S frame to the frame S' moving with the conductor (assume $x = x'$ at $t = t' = 0$)? Then, rewrite the argument of the cosine factor in Eq. (1) in terms of the coordinates x' and t' in the S' frame.

(b) **[3 points]** From the results of Part (a), define the wavevector k' and frequency ω' of the incident EM wave in the S' frame and express them in terms of k and ω. Then, using the relation $\omega = ck$, express k' in terms of k and, similarly, ω' in terms of ω.

(c) **[3 points]** How do the amplitudes of the electric and magnetic fields of the wave in Eq. (1) transform when going to the S' frame? Express your answers in terms of E_0.

Combining with the results of Part (b), write down the fields $\boldsymbol{E}'(x',t')$ and $\boldsymbol{B}'(x',t')$ of the incident wave in the S' frame.

(d) **[4 points]** The wave reflects completely from the ideal conductor. What are the wavenumber $\widetilde{k}'$ and frequency $\widetilde{\omega}'$ of the reflected wave in the S' frame in terms of k' and ω' of the incident wave in this frame?

What is the relation between the fields $\boldsymbol{E}'$ and $\boldsymbol{B}'$ of the incident wave and those of the reflected wave, $\widetilde{\boldsymbol{E}'}$ and $\widetilde{\boldsymbol{B}'}$, at the surface of the ideal conductor?

Then obtain the fields $\widetilde{\boldsymbol{E}'}(x',t')$ and $\widetilde{\boldsymbol{B}'}(x',t')$ of the reflected wave in the S' frame.

(e) **[4 points]** By transforming back to the S frame, find the fields $\widetilde{\boldsymbol{E}}(x,t)$ and $\widetilde{\boldsymbol{B}}(x,t)$ of the reflected wave in the laboratory frame.

What are the wavenumber $\widetilde{k}$ and frequency $\widetilde{\omega}$ of the reflected wave in terms of k and ω of the incident wave?

(f) **[1 point]** Is the frequency of the reflected wave $\widetilde{\omega}$ higher or lower than ω of the incident wave for the cases of $v > 0$ and $v < 0$ (i.e., when the conductor is moving away from or towards the source of the incident wave, respectively)?

(g) **[1 point]** A radar of frequency ω is used to determine the speed of a car by measuring the frequency $\widetilde{\omega}$ of the EM wave reflected from it. Assuming the car moves along the line of sight, what is its speed in terms of ω and $\widetilde{\omega}$ in the non-relativistic limit $\beta \ll 1$?

(h) **[6 points]** Using the waveforms from the previous parts, calculate the force per unit area, i.e., pressure P, exerted on the moving conductor in the laboratory frame S.

Express your answer for $P(\beta)$ in terms of the pressure P_0 for $\beta = 0$. Discuss the limits $\beta \to \pm 1$ for $P(\beta)$.

Possibly useful formulae: $\boldsymbol{E}'_\perp = \frac{1}{\sqrt{1-\beta^2}}\left[\boldsymbol{E}_\perp + (\boldsymbol{v} \times \boldsymbol{B}_\perp)\right]$; $\boldsymbol{B}'_\perp = \frac{1}{\sqrt{1-\beta^2}}\left[\boldsymbol{B}_\perp - \frac{1}{c^2}(\boldsymbol{v} \times \boldsymbol{E}_\perp)\right]$

Problem B.3

A hydrogen atom is in its ground state $n = 1$ with energy E_1. An electric field $\mathscr{E}(t)$ pointing along the z axis is turned on at the time $t = 0$ and then exponentially decreases in time t:

$$\mathscr{E}(t) = \begin{cases} 0 & \text{for} \quad t < 0, \\ \mathscr{E}_0 \, e^{-t/\tau} & \text{for} \quad t > 0, \quad \mathscr{E}_0 \| \boldsymbol{z} \end{cases} \tag{1}$$

Assume the proton is stationary, while the electron experiences the perturbation $V(t) = -qz\mathscr{E}(t)$, where q is the charge, and z is the electron coordinate relative to the proton.

(a) **[4 points]** Using symmetry arguments, determine which matrix elements of the electron coordinate z vanish identically between the ground state and the four excited states $|n, l, m\rangle$ with $n = 2$: $|2, 0, 0\rangle$, $|2, 1, +1\rangle$, $|2, 1, -1\rangle$, and $|2, 1, 0\rangle$.

(b) **[5 points]** Using the hydrogen atom wavefunctions, calculate the nonvanishing matrix element (or elements) among those considered in Part (a). Represent your result in the form Ba, where a is the Bohr radius, and B is a numerical coefficient.

(c) **[5 points]** To the first order of *time-dependent* perturbation theory in $V(t)$, compute the amplitudes and the probabilities for the atom to be found at $t \to \infty$ in each of the four excited states $|2, l, m\rangle$ with $n = 2$, utilizing the results of Parts (a) and (b).

(d) **[3 points]** Now suppose τ is sufficiently long, so that $\hbar/\tau \ll E_2 - E_1$. From Part (c), what is the probability of exciting the atom from $n = 1$ to $n = 2$ in this limit?

In the next two parts, show that the probability found in Part (d) can be also obtained in an alternate way as a combination of a *sudden* perturbation, followed by an *adiabatic* evolution.

(e) **[5 points]** When τ is long, the electric field in Eq. (1) is approximately constant for $0 < t \ll \tau$. Using *time-independent* perturbation theory, find how the states $n = 1$ and $n = 2$ are modified due to this electric field.

Then, using the *sudden* approximation at $t = 0$ (where $\mathscr{E}(t)$ changes discontinuously), obtain the probability of transition to the first excited state from $t = 0^-$ to $t = 0^+$.

(f) **[3 points]** The sudden perturbation at $t = 0$ is followed by *adiabatic* evolution on the time scale $t \gg \tau$. How does the probability of transition to the first excited state change during this time? Does your result at $t \to \infty$ agree with the result of Part (d)?

The hydrogen atom wavefunctions have the form $\psi_{nlm} = R_{nl}(r)\, Y_l^m(\theta, \phi)$, where

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta, \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\phi} \sin\theta, \tag{2}$$

$$R_{10} = \frac{2}{a^{3/2}}\, e^{-r/a}, \qquad R_{20} = \frac{2 - r/a}{(2a)^{3/2}}\, e^{-r/2a}, \qquad R_{21} = \frac{1}{(2a)^{3/2}} \frac{r}{\sqrt{3}a}\, e^{-r/2a}. \tag{3}$$

Possibly useful integral: $\int_0^\infty x^n e^{-x/b} dx = n!\, b^{n+1}$.

Problem B.4

A beam of electrons of mass m and energy $E = \hbar^2 k^2/2m$ scatters off a molecule consisting of two identical atoms separated by the distance a. The molecule has a fixed position in space with a fixed vector $\boldsymbol{a}$ connecting the two atoms. Suppose one atom produces a spherically-symmetrical potential $U_0(r)$, where r is the distance from the center of the atom. Then, the potential produced by the two atoms in the molecule is

$$U(\boldsymbol{r}) = U_0(r) + U_0(|\boldsymbol{r} - \boldsymbol{a}|). \tag{1}$$

Assume that the scattering amplitude $f_0(q)$ on the single-atom potential $U_0(r)$ in the Born approximation is known

$$f_0(q) = -\frac{m}{2\pi\hbar^2} \int e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\, U_0(r) d^3r, \tag{2}$$

where $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$ is the change of the electron wavevector from $\boldsymbol{k}$ to $\boldsymbol{k}'$ upon scattering, $q = |\boldsymbol{k}' - \boldsymbol{k}| = 2k\sin(\theta/2)$ and θ is the scattering angle.

(a) **[4 points]** Obtain a formula for the scattering amplitude $f(\boldsymbol{q})$ on the two-atom potential (1) in the Born approximation in terms of $\boldsymbol{a}$ and $f_0(q)$ in Eq. (2).

(b) **[4 points]** Express the differential cross-section of scattering on the molecule, $d\sigma/d\Omega$, in terms of the corresponding cross-section $d\sigma_0/d\Omega$ for a single atom in the Born approximation.

(c) **[4 points]** In this Part only, suppose that $\boldsymbol{a}$ is perpendicular to the incoming $\boldsymbol{k}$ and only consider $\boldsymbol{k}'$ in the plane of $\boldsymbol{k}$ and $\boldsymbol{a}$. Assuming $ka \geq \pi$, determine the scattering angles θ_n where $d\sigma/d\Omega$ vanishes. (Assume that $d\sigma_0/d\Omega$ does not vanish for any θ.)

(d) **[3 points]** Now assume that electrons have low energy, so that $ka \ll 1$, while the Born approximation is still applicable. Express $d\sigma/d\Omega$ in terms of $d\sigma_0/d\Omega$ in this limit. What is the ratio of the total cross sections σ^{tot} and σ_0^{tot} in this limit? Interpret the results physically.

(e) **[3 points]** Now consider the high-energy limit where $ka \gg 1$. Express $d\sigma/d\Omega$ in terms of $d\sigma_0/d\Omega$ in this limit. What is the ratio of the total cross sections σ^{tot} and σ_0^{tot} in this limit? Interpret the results physically. (If necessary you may assume that the range of the potential $U_0(r)$ is much shorter than the interatomic distance.)

(f) **[7 points]** The orientation of the molecular axis is determined by the unit vector $\hat{\boldsymbol{a}} = \boldsymbol{a}/a$. So far, we assumed that the vector $\hat{\boldsymbol{a}}$ has a fixed orientation. Now assume that the molecular axis orientations are random and appear with equal probability. Average the differential cross-section of scattering over random orientations of $\hat{\boldsymbol{a}}$ and obtain the averaged cross-section $\overline{d\sigma/d\Omega}$. Briefly explain how the obtained result can be used to determine the interatomic distance a from scattering data.