# UNIVERSITY OF MARYLAND
# Department of Physics
# College Park, Maryland

# PHYSICS Ph.D. QUALIFYING EXAMINATION
# PART I

**January 21, 2021 9:00 a.m. – 1:00 p.m.**

**Do any four problems. Each problem is worth 25 points.**
**Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).**

**Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)**

**At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.**

**You may keep this packet with the questions after the exam.**

## Problem I.1

In 1896, Zeeman first observed the effect of a magnetic field on spectral lines. Lorentz proposed a simple (classical) model to interpret these observations. Though very far from the quantum truth of the matter, this model was useful. In this problem you'll redo some calculations with this model.

(a) **[6 points]** Using either $\vec{F} = m\vec{a}$ or the Lagrangian method, find the equations of motion for a particle of mass $m$ and charge $q$ in a three dimensional harmonic oscillator potential of natural angular frequency $\omega_0$, in the presence of a uniform magnetic field $\vec{B} = B\hat{z}$, with $B > 0$. You may use either Cartesian coordinates $(x, y, z)$ or cylindrical coordinates $(\rho, \phi, z)$ for the particle's position.

(b) **[7 points]** Find exact expressions for all the positive normal mode frequencies describing the charge's three-dimensional motion, and expand them to linear order in the magnetic field.

(c) **[2 points]** Lorentz assumed that the frequency of a normal mode corresponded to the frequency of light emitted by the atom. On that assumption, if $\Delta\omega$ is the observed difference between the highest and lowest frequencies among the split components of a spectral line in the presence of the magnetic field $B$, what charge to mass ratio $|q|/m$ could be inferred for the radiating particle in this model? (Assume the magnetic field has a very weak effect on the frequencies, so you can use your result from part (b) to linear order in $B$.)

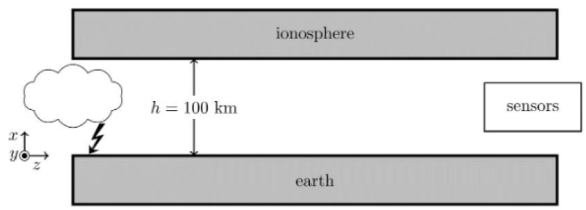
(d) **[6 points]** Find the time dependence of the oscillator coordinates for each of the normal mode solutions.

(e) **[4 points]** What does this model predict for the polarization of the highest frequency component of a split spectral line, for light emitted in the direction of the magnetic field? How could the *sign* of the radiating charge be inferred from observations of this polarization?

## Problem I.2

A lightning discharge generates a broad spectrum of electromagnetic (EM) waves at frequencies of a few Hz to tens of kHz. The emitted EM waves propagate to the far field guided by the conducting earth (mostly seawater) and the ionosphere above the earth at a height $h \approx 100$ km (see figure below).

We can model this scenario as a parallel-plate waveguide with infinitely large width in the $y$ direction, so that any fringe effects can be ignored and the wave fields can be taken to be independent of $y$. We also assume that the plates are perfect conductors ($\sigma = \infty$ ) and that the medium in between is vacuum. We use this model to explore the properties of the EM wave propagation along the $z$-direction, detected by sensors in the far field many wavelengths away from the discharge.




(a) [**5 points**] Consider TE modes (i.e. $\mathbf{E} = E_y \hat{y}$) propagating to the right, in the $\hat{z}$ direction. What are the components of $\mathbf{B}$ associated with this mode? Sketch the lowest order mode, including $\mathbf{E}$ and $\mathbf{B}$, and indicate the direction of the Poynting flux.

(b) [**5 points**]

Starting from the wave equation for $E_y$, derive the dispersion relation ($k_z$ versus angular frequency $\omega$). Is there any cut-off frequency, $f_c$, below which no TE modes propagate? If so, find $f_c$. (The letter $f$ denotes frequency in Hz, i.e. cycles/s.)

(c) [**5 points**] Consider a TM mode (i.e. $\mathbf{B} = B_y \hat{y}$), again propagating to the right. What are the non-zero components of $\mathbf{E}$ associated with this mode? Starting from the wave equation for $B_y$, find the dispersion relation. Is there any cut-off frequency $f_c$ below which no TM modes propagate? If so, find $f_c$.

(d) [**5 points**] A wave launched by a lightning strike is dispersive, so different frequency components arrive at the sensors at different times. If the sensors are located at a distance $d$ from the lightning strike, when does a wavepacket component with frequency $f$ arrive at the sensors? Assume a dispersion relation of the general form $c^2 k_z^2 = \omega^2 - \omega_c^2$.

(e) [**5 points**] In reality the conductivities of sea water and the ionosphere are finite. Namely, $\sigma_{\text{seawater}} \approx 4 (\Omega\text{m})^{-1}$ and $\sigma_{\text{ionosphere}} \approx 10^{-4} (\Omega\text{m})^{-1}$. Estimate how good our perfect conductor assumption is for the frequency $f = 100$ Hz.

<u>*Possibly useful information*</u>:

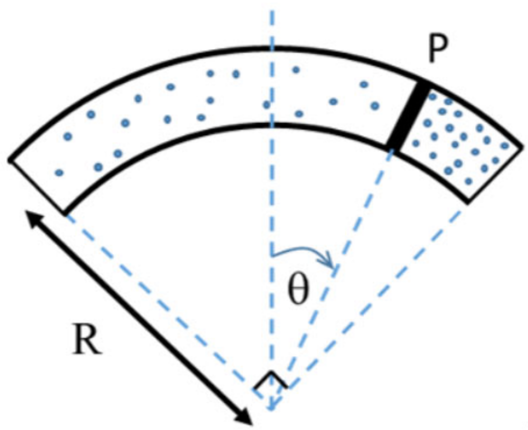
$c = 1/\sqrt{\epsilon_0 \mu_0} = 3 \times 10^8$ m/s

$\epsilon_0 = 8.8 \times 10^{-12}$ F/m

skin depth for a good conductor: $\delta = \sqrt{2/\mu\sigma\omega}$

## Problem I.3

The sketch below shows an airtight tube of cross-sectional area $A$, bent in a circular arc of radius $R$ and having a total angular spread of $\pi/2$ radians. The radial width of the tube interior is much less than $R$. In the tube is a piston (marked P in the sketch) of mass $M$, as well as negligible thickness, is acted on by gravity. The angular position of the piston is $\theta$, measured from the vertical dashed line. The dots represent *ideal-gas* monomers (e.g., neon atoms); the number on the left side of the piston ($N_L$) and on the right side ($N_R$) are equal: $N_L = N_R = N_0$. The entire system is enclosed in a heat bath at temperature $T$. For high $T$, the high pressures on both sides of the piston force it to the equilibrium position $\theta = 0$. The theme of this problem is to explore what happens as $T$ decreases.




(a) **(i)** [**7 points**] Find the net [total] force on the piston. For small displacements $\theta$ from $\theta = 0$, show that to leading order in $\theta$ the net force on the piston has the Hooke's Law form $F = -k_H R\theta$ and determine $k_H$. (The factor of $R$ is pulled out explicitly so that $k_H$ has the usual units of force per length.) Neglect the effects of gravity on the gas [single-atom] molecules, any atom-boundary interactions, and friction between the piston and the bent tube.

**(ii)** [**2 points**] Find the [angular] oscillation frequency $\omega$ of $\theta$ about $\theta = 0$.

(b) **(i)** [**3 points**] For what temperature $T_0$ do $k_H$ and $\omega$ vanish?

**(ii)** [**2 points**] For $T \lesssim T_0$, what happens, qualitatively?

(c) **(i)** [**1 point**] If we extend the expansion in part (a)(i) beyond linear order in $\theta$, do you expect all powers of $\theta$ or just odd powers, or just even powers, to be present? Very briefly justify your choice.

**(ii)** [**7 points**] For $T \lesssim T_0$, determine the new equilibrium position[s] of $\theta$ by expanding the force-balance equation to the next non-vanishing order of $\theta$. Express your answer as a function of $T_0 - T$ or the dimensionless variable $t \equiv 1 - (T/T_0)$, i.e. $T = T_0(1 - t)$; show clearly the lowest-order dependence of $\theta$ on $t$ or $T_0 - T$.

**(iii)** [**1 point**] For $T \lesssim T_0$, is $\theta = 0$ still a solution? Is it stable?

(d) [**2 points**] The angle $\theta$ is in many ways analogous to a magnetization. Propose a modification of this piston-in-bent-tube system that would introduce an effect analogous to that of an external magnetic field in a magnetic system.

## Problem I.4

An electron and a positron collide head on, producing a resonance called $\Upsilon(4s)$, which breaks up into a $B^+$ meson and a $B^-$ meson and no other particles. The $\Upsilon(4s)$ rest energy is $M_{\Upsilon(4s)} = 10.579\,\mathrm{GeV}/c^2$, and the rest mass of each $B$ meson is $m_B = 5.279\,\mathrm{GeV}/c^2$. The $B$ mesons subsequently decay into other particles, and by studying how the decay products depend on the locations of the decay vertices, CP violation can be measured. To spread out the decay locations, collisions with unequal electron and positron energies are studied.

(a) [**10 points**] If the electron has energy $E_- = 9$ GeV in the lab frame, what should be the positron energy $E_+$, in order to produce the $\Upsilon(4s)$ state at threshold, i.e. with no energy to spare?

(b) [**4 points**] What is the speed of the $B$ mesons in the rest frame of the $\Upsilon(4s)$?

(c) [**1 points**] The mean lifetime of the B mesons is $\tau = 1.64$ps (1 ps $= 10^{-12}$ s). If the $B^+$ decays after 3 ps and the $B^-$ decays after 1 ps, how far apart are their decay vertices in the rest frame of the $\Upsilon(4s)$?

(d) [**10 points**] If the $B^+$ in the previous part travels in the same direction as the original electron, and the $B^-$ travels in the opposite direction in the rest frame of the $\Upsilon(4s)$, how far apart are the decay vertices in the lab frame? (The distance may seem small, but vertex separations of 100 microns or even smaller can sometimes be resolved in the experiment.)

*Possibly useful information*:
$c = 3 \times 10^8$ m/s
1 ps $= 10^{-12}$ s
It is recommended to adopt units with $c = 1$ in most calculations here.
The electron mass is $0.511\mathrm{MeV}/c^2$, which is small enough to be neglected here.
You may use a nonrelativistic approximation in one part of this problem.

**Problem I.5**

The motion of gas with planar symmetry satisfies the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial x}, \tag{2}$$

$$p = \rho k_B T/m, \tag{3}$$

where $\rho(x,t)$ is the mass density, $v(x,t)$ is the gas velocity in the $x$ direction, $p(x,t)$ is the pressure, $T(x,t)$ is the temperature, $k_B$ is Boltzmann's constant, $m$ is the mass of a gas molecule, and $t$ is time.

(a) **[9 points]**

Assume that the connection between the pressure and density is well approximated by the adiabatic relationship,

$$p\rho^{-\gamma} = \mathit{constant}, \tag{4}$$

where $\gamma$ is the ratio of specific heats. [Eq. (4) is justified if heat flow due to thermal conductivity of the gas can be neglected.] Consider small-amplitude perturbations of a static, homogeneous background,

$$\begin{aligned} \rho &= \rho_0 + \delta\rho(x,t), \\ v &= \delta v(x,t), \\ p &= p_0 + \delta p(x,t), \\ T &= T_0 + \delta T(x,t), \end{aligned}$$

where $\rho_0$, $p_0$ and $T_0$ are background values that are independent of $x$ and $t$, and $\delta\rho$, $\delta v$, $\delta p$, $\delta T$ are the perturbations, which are "small" in the sense that it is a good approximation to expand the equations (1)-(3), keeping only terms that are linear in these perturbations. Show that the perturbed density satisfies the wave equation,

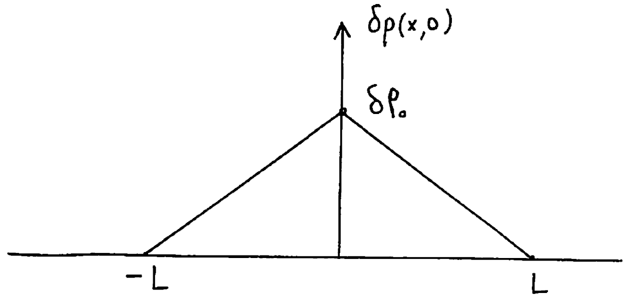
$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_s^2 \frac{\partial^2 \delta\rho}{\partial x^2} = 0, \tag{5}$$

and give an expression for the sound speed $c_s$.

(b) **[1 point]** Assuming that $\delta\rho \ll \rho_0$, $\delta p \ll p_0$ and $\delta T \ll T_0$, what is the condition on $\delta v$ in order for it to be a good approximation to keep only terms that are linear in the perturbations?

(c) **[5 points]** What is the most general solution of the wave equation (5) for $\delta\rho(x,t)$?

(d) **[5 points]** Assume that at $t = 0$, $\delta v(x,0) = 0$, and $\delta\rho(x,0)$ is as shown in the figure below. Draw a sketch of $\delta\rho(x,t)$ at times $t = t_1 \equiv L/2c_s$ and $t = t_2 \equiv 2L/c_s$. Be sure to label all significant values on the horizontal and vertical axes in both sketches.




(e) [**5 points**] Isaac Newton obtained the result $\sqrt{p_0/\rho_0}$ for the speed of sound, under the assumption that the gas obeyed an isothermal relationship between pressure and density (rather than the adiabatic one (4)). Newton's result does not agree with measured values for typical gases at room temperature and pressure. Let $\kappa$ denote the thermal diffusivity of the gas — i.e., the temperature satisfies the diffusion equation $\partial T/\partial t = \kappa\, \partial^2 T/\partial x^2$ in the case $v = 0$. Using simple estimates, give an inequality (involving the symbol $\ll$) specifying how small $\kappa$ must be for a sound wave of wavelength $\lambda$ to be accurately described by the adiabatic relation.