UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

January 22, 2021 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II-1

Consider the bound state of a heavy quark (q) and its antiquark ($\bar{q}$) (quarkonium) in a linearly rising potential given by $U(r) = \sigma r$. The heaviness of the quarks (e.g. charmed) allows the use of non-relativistic quantum mechanics. Express the *s-wave* radial wave function in the form $\psi(r) = \frac{\phi(r)}{r}$.

(a) [**4 points**] Write down the radial Schrödinger equation for $\phi(r)$.

(b) [**5 points**] The Airy function Ai(x) satisfies the equation $\left[\frac{d^2}{dx^2} - x\right] \mathrm{Ai(x)} = 0$ and has its zeros at $x = -2.338, -4.008, -5.521$, etc. As $x \to \infty$, $\mathrm{Ai(x)} \to x^{-1/4} \exp\left(-\frac{2}{3} x^{3/2}\right)$. By a suitable choice of variables, cast the Schrödinger equation for $\phi(r)$ into the form of the Airy equation.

(c) [**7 points**] Obtain an expression for the lowest bound-state energy in terms of the reduced Planck's constant ($\hbar$), the quark mass (m_q), and the string constant (σ) on the wave function obtained from the Airy equation by imposing appropriate boundary conditions on the wave function obtained from the Airy equation.

(d) [**7 points**] Consider treating the problem of the ground state of the system by using the Rayleigh-Ritz variational principle. Examine the following two possible choices for the radial wavefunct1on.

(1) $\phi_1(r) = N_1 \, r \exp(-r/a)$, where N_1 la a normalization constant which ensures that $\int (\phi_1(r))^2 \, dr = 1$. The expectation value of H, the quarkonium hamiltonian in ϕ_1 is

$$(\phi_1, \, H \, \phi_1) = \frac{\hbar^2}{m_q a^2} + \frac{3}{2} \sigma \, a$$

(2) $\phi_2(r) = N_2 \, r \exp(-r^2/b^2)$, where N_2 is a normalization constant which ensures that $\int (\phi_2(r))^2 \, dr = 1$. The corresponding expectation value of H is

$$(\phi_2, \, H \, \phi_2) = \frac{3\hbar^2}{2 m_q b^2} + \frac{2}{\sqrt{\pi}} \sigma \, b$$

By carrying out the variational calculation, establish which one of the two is the better approximation.

(e) [**2 points**] Give a brief physical explanation why one choice gives lower energy than the other for the result you obtained in (d).

Problem II.2

A particle of mass m in one spatial dimension moves in an attractive potential $U(x) = -\lambda\delta(x)$, where $\delta(x)$ is the Dirac delta function, and $\lambda > 0$. There is one bound state, with normalized wave function

$$\psi_0(x) = \sqrt{\kappa}e^{-\kappa|x|}, \tag{1}$$

and the bound state energy is $E_0 = -\hbar^2\kappa^2/2m = -m\lambda^2/2\hbar^2$.

At time $t < 0$ the particle is in the bound state. At time $t = 0$ a small ac electric field $\mathcal{E} = \mathcal{E}_0 \sin(\omega t)$ with angular frequency $\omega > |E_0|/\hbar$ is turned on. The Hamiltonian of the perturbation is

$$V = -q\mathcal{E}_0 x \sin(\omega t),$$

where q is the electric charge of the particle. This perturbation may cause a transition from the bound state to unbound states (analogous to ionization of an atom).

(a) [**5 points**] Why are there first order transitions only to odd parity states?

(b) [**4 points**] Find the wave functions and energies of the unbound states that are odd under the parity operation $x \to -x$. For normalization, assume the particle is confined to a box extending from $x = -L$ to $x = L$.

(c) [**2 points**] Calculate the nonzero matrix elements for first order transitions from the ground state to an unbound state. Assume that the box size L is much greater than the extent of the bound state, so that the limits of x integration can be taken to infinity when evaluating the matrix elements.

(d) [**10 points**] Using the Fermi Golden Rule, calculate the ionization rate of this analog "atom", i.e. the probability per unit time for a transition of the particle from the ground state to an unbound state. (*Tip*: Check the dimensionality of your final result.)

(e) [**4 points**] Sketch how the ionization rate Γ depends on the frequency ω. What is the power law dependence of Γ on frequency near the threshold for ionization?

<u>*Useful integral*</u>: $\int_0^\infty dx\, x \sin(ax)e^{-bx} = \frac{2ab}{(a^2+b^2)^2}$

Problem II.3

Consider the scattering of low energy neutrons from protons. The nuclear potential between these nucleons may be approximated by $V = -V_0$ for $r \leq a$ and $V = 0$ for $r > a$, where $V_0 = 23$ MeV and $a = 2$ fm. Assume that the neutron and proton masses are both equal to 940 MeV/c^2, and take $\hbar c = 200$ MeV-fm (1 fm $= 10^{-13}$ cm).

(a) [**3 points**] Give a general definition of the differential cross section.

(b) [**5 points**] Show that, for incident neutrons with laboratory kinetic energy $E_{\text{lab}} = 1$ MeV, it is reasonable to consider only s-wave scattering in this process.

(c) [**5 points**] State the two alternative sufficient conditions for validity of the Born approximation, and check whether or not the Born approximation is valid for this process.

(d) [**6 points**] Find a transcendental equation that determines the s-wave phase shift for this scattering process. To save time, you need not evaluate the result numerically.

(e) [**3 points**] Write an expression for the s-wave total cross section in terms of the s-wave phase shift.

(f) [**3 points**] Find an expression for the total cross section in the limit of zero neutron energy, and evaluate it numerically.

Possibly useful information: The wave function for a scattered particle has the asymptotic form

$$\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\theta, \phi)\frac{e^{ikr}}{r}$$

For scattering from a spherical potential, the partial wave expansion for the scattering amplitude $f(\theta)$ is

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)(\sin\delta_\ell)e^{i\delta_\ell}P_\ell(\cos\theta)$$

where $P_\ell(\cos\theta)$ is the Legendre polynomial, which satisfies $P_\ell(1) = 1$.

Problem II.4

Consider a neutral carbon-12 atom ($Z = 6$).

(a) **[2 points]** Write out the configuration (in terms of single particle orbitals) of the atomic electrons in the ground state.

(b) **[6 points]** Assuming that four of the electrons are in s orbitals and have total angular momentum values $L = S = J = 0$, what are the possible angular momentum states of the atom *ignoring* the Pauli principle for the two other electrons? Use the term symbol notation ${}^{(2S+1)}L_J$ to denote the states. (The nuclear spin of carbon-12 is zero.)

(c) **[6 points]** Which states are allowed after imposing the Pauli principle?

(d) **[5 points]** What are the total spin and orbital angular momentum of the ground state? Why does this value of the total spin give the lowest energy? What is the parity of the ground state?

(e) (i) **[2 points]** The atom captures a negative muon which replaces one of the electrons, making a neutral muonic carbon atom. After this system fully settles down, what is the ground state configuration (in terms of single particle orbitals)?

(ii) **[4 points]** Give a rough estimate of the binding energy of the muon in the new ground state in electron volts. (The mass of a muon is 105.7 MeV/c^2.)

Problem II.5

A narrow wire of length L can be treated as a one-dimensional potential-free region. The electrons in the wire are allowed to come to equilibrium with a reservoir of electrons in a large piece of metal at chemical potential μ and at temperature T. There can be a net transfer of electrons between the wire and the metal, until thermal equilibrium is reached. You may ignore the (electrostatic) interactions between the multiple electrons that fill the wire. Let k_B denote the Boltzmann constant as usual.

(a) [**4 points**] To find the allowed solutions of the Schrödinger equation, make the approximation that the electrons of mass m which occupy the wire are confined between two walls of infinite potential at $x = 0$ and $x = L$. Find these energies ϵ_n, where n is a positive integer. Does the spacing increase, decrease, or stay the same as n (and so energy) increase?

(b) [**3 points**] To account for the equilibrium properties of the electrons at specified μ and T, we can use classical mechanics at high temperatures, but must use quantum mechanics at low temperatures. Write down an inequality involving the parameters of the problem that determines when the statistical physics of the system cannot be considered classical.

Henceforth, we will be working in the low-temperature region of the quantum regime, assuming that the inequality is a very strong inequality.

(c) [**5 points**] Now consider the limit in which $\mu \gg \epsilon_1$, where ϵ_1 is the lowest eigenenergy, such that average number of electrons $N(\mu, T)$ filling the wire is large. Find whether $N(\mu, T)$ increases, decreases, or stays the same as T begins to increase from $T = 0$, keeping μ fixed. You may work analytically, starting with the quasicontinuum density of states, or use graphical methods without equations, qualitatively indicating how the spacings between states changes with n along with how the Fermi-Dirac distribution (FDD)* changes with energy. Explain your reasoning clearly but succinctly.

**Reminder*: The FDD is 1/2 at $\epsilon = \mu$ and is an odd function in $(\epsilon - \mu)$ about that point.

(d) [**5 points**] Under these same assumptions, show that the number of electrons in the wire as a function of μ and T is given by:

$$N(\mu, T) = A_N \int_{-\alpha}^{\infty} \frac{ds}{(1 + e^s)(\alpha + s)^{1/2}} \qquad \text{where} \qquad \alpha \equiv \frac{\mu}{k_B T} \tag{1}$$

and A_N is a constant for which you should give an explicit expression.

(e) [**8 points**] Again at fixed μ, write an expression for the internal energy $U(\mu, T)$ as well as the heat capacity $C_\mu(T)$. How does $C_\mu(T)$ depend on T at low temperatures?