UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART I

January 24, 2019 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

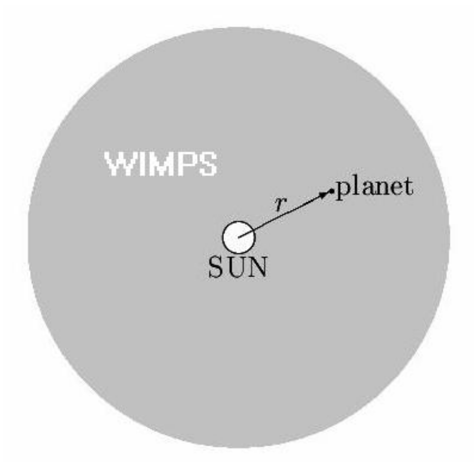
Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1

A solar system is immersed in a uniform spherical cloud of Weakly-Interacting Massive Particles (WIMPs) of mass density ρ and radius R_W. The sun is at rest at the center of the cloud. A planet of mass m is located at a radius r from the sun. Assume that the planet and the Sun can be considered as point particles of mass m and $M_\odot$ respectively, with $M_\odot \gg m$.



(a) **[3 points]** Show that the force on the planet can be written as

$$\vec{F} = -m\left(\frac{k}{r^2} + br\right)\hat{r},$$

where $\hat{r}$ is a unit vector in the radial direction. Express the constants k and b in terms of $M_\odot$, ρ and G (Newton's gravitational constant).

(b) **[3 points]** Find a potential energy $V(r)$ associated with the conservative force $\vec{F}$, valid within the WIMP cloud (i.e., for $r < R_W$). You may use k and b here and in later parts.

(c) **[2 points]** Because of the spherical symmetry, the planet must orbit in a plane. Write the Lagrangian L of the planet, using the distance r and azimuth angle θ as generalized coordinates.

(d) **[4 points]** From the Lagrangian, derive the canonical momenta conjugate to r and θ and obtain the Hamiltonian of the planet.

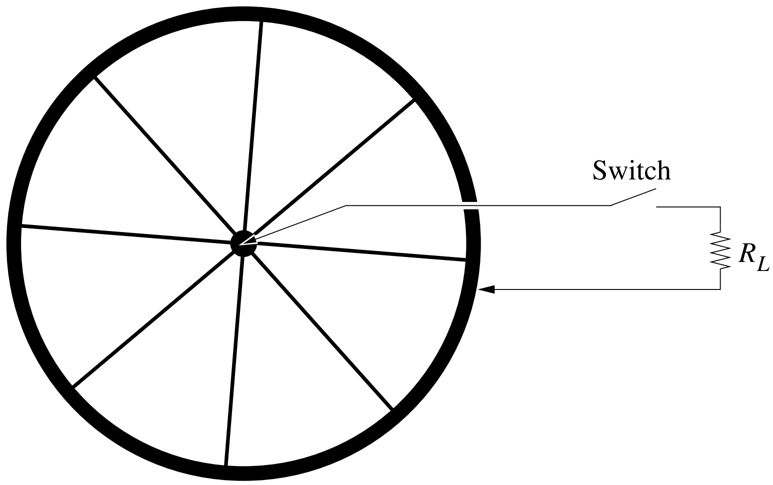
(e) **[3 points]** Using the symbol ℓ to represent the canonical momentum conjugate to θ, reduce the Hamiltonian to that for a particle moving in an effective one-dimensional potential energy $V_{\text{eff}}(r)$, and find $V_{\text{eff}}(r)$.

(f) **[3 points]** It is observed that the planet moves in a circular orbit with radius $r = r_0$. Find an algebraic expression that relates the radius r_0, the force constants k, b, and the conserved quantity ℓ (do not solve for r_0).
Find the angular frequency $\dot{\theta}$ in terms of k, b and r_0.

(g) [**4 points**] Now consider a planet on a *nearly* circular orbit $r \approx r_0$. Its radial motion is an oscillation about the circular orbit. Find the angular frequency ω of this small-amplitude oscillation.

(h) [**3 points**] Based on the findings from above, describe how observations of a planet's orbit can be used to test the hypothesis that there is a cloud of WIMPs around the sun.

Problem I.2



A flywheel with thin spokes, as shown in the figure, has radius a and moment-of-inertia I. Each radial spoke of the wheel is an electrical conductor with resistance R_S along its length, while the central hub and the circular rim are good conductors with negligible resistance. There is a uniform magnetic field $\boldsymbol{B}$ directed **into the page**, normal to the plane of the wheel. The wheel is set into rotation on frictionless bearings at angular velocity ω_0 and then allowed to coast.

Leads are placed in frictionless contact with the center and the rim of the wheel, and a circuit with a switch is set up as shown, initially open. There is a load resistance R_L in the external part of the circuit.

Address each of the questions below with your answers given in terms of ω_0 or ω, a, I, B, R_S and R_L.

(a) **[7 points]** Considering mechanical equilibrium of a charged particle on a rotating spoke, determine the electric potential difference across the open switch.

(b) **[5 points]** The switch is now closed, allowing current i to flow. At an arbitrary rotation rate ω, calculate the current i flowing through the switch and the total power dissipated in all parts of the circuit, P.

(c) [**2 points**] If the wheel is rotating counter-clockwise, which direction does the current flow: outward (from hub to rim) or inward (from rim to hub)?

(d) [**4 points**] With the switch closed and the wheel rotating at ω, calculate the torque on the wheel arising from the current flowing radially through the spokes.

(e) [**3 points**] With the switch closed, what is the angular acceleration rate of the wheel at instantaneous rotation rate $\omega(t)$?

(f) [**4 points**] Find $\omega(t)$ for $t > 0$ if the switch is closed at $t = 0$.

Problem I.3

Consider a binary alloy where each site of a lattice is occupied by an atom of type A or B. (A realistic alloy might mix roughly half copper and half zinc to make $\beta-$brass.) Let the numbers of the two kinds of atoms be N_A and N_B, with $N_A + N_B = N$ (i.e., the total number of sites is fixed at N). Define the concentrations $n_A \equiv N_A/N$ and $n_B \equiv N_B/N$, and the difference $x \equiv n_A - n_B$. The interaction energies between the neighboring atoms of the types AA, BB, and AB are ε_{AA}, ε_{BB}, and ε_{AB}, correspondingly. Let c denote the number of nearest neighbors for each atom.

(a) [**1 point**] For a cubic lattice in three dimensions, what is c?
(But use the symbol c in calculations in the rest of this problem, not this number.)

(b) [**4 points**] Consider the system at a high enough temperature such that the atoms are randomly distributed among the sites. Calculate the average interaction energy U per site under these conditions, first expressing U in terms of n_A and n_B, and then substitute to obtain $U(x)$.

(c) [**4 points**] From now on, for simplicity, assume that $\varepsilon_{AA} = \varepsilon_{BB} = \varepsilon_0$ and $\varepsilon_{AB} > \varepsilon_0$. Obtain $U(x)$ in this this case and sketch a plot of it for $-1 \leq x \leq 1$. Indicate values of $U(x)$ at its extrema.

(d) [**6 points**] Under the same conditions (where the atoms are randomly distributed among the sites), calculate the configurational entropy S per site. Assume that $N_A, N_B \gg 1$, so the Stirling approximation $\ln(N!) \approx N \ln N - N$ can be used. First express S in terms of n_A and n_B, and then obtain $S(x)$.

Sketch a plot of the function $S(x)$, and indicate the values of $S(x)$ at its extrema.

(e) [**5 points**] Using the results from the previous parts, obtain the free energy per site $F(x,T) = U(x) - TS(x)$, where T is the temperature

Show that, at a high temperature, $F(x)$ has one global minimum as a function of x. Show that, at a low temperature, $F(x)$ has one local maximum surrounded by two minima, excluding the boundaries at $x = \pm 1$.

(f) [**5 points**] A binary alloy may be stable in a *mixed* state, in which the atoms are randomly distributed among the sites with the same x throughout, or it may be more favorable to spontaneously separate into two phases (an *unmixed* state) with different values of x, if such a segregation decreases the free energy F. For our two-component system, if $x = 0$, which state is favored at high temperature, and which is favored at low temperature?

Calculate the temperature T_* at which the transition from mixed to unmixed occurs. (Hint: The system remains stable in the mixed state as long as $d^2F/dx^2 > 0$ for all x.)

Problem I.4

Ultra-high-energy cosmic ray protons can lose energy by inelastic collisions with cosmic microwave background (CMB) photons, producing pions: In this problem, we will only look at the reaction $p + \gamma \to p + \pi^0$.

(a) [**8 points**] First, consider this reaction in the reference frame in which the proton is initially at rest. What is the minimum ("threshold") energy, E_t that the photon must have to produce a π^0 by this reaction? (Express your answer in terms of m_p and m_π, the masses of the proton and the pion, respectively.)
Hint: at threshold there is no kinetic energy in the final-state center-of-mass frame.

(b) [**3 points**] Qualitatively, how does that threshold photon energy compare to the pion and/or proton rest energy?

(c) [**3 points**] What is the approximate energy of a typical CMB photon? (Hint: use what you know about the temperature of the CMB, perhaps.)
For comparison, a neutral pion has a mass of $> 100\,\mathrm{MeV/c^2}$.

(d) [**8 points**] Now, consider the same reaction as in part (a) but viewed in the reference frame of the interstellar medium, in which the proton collides with a CMB photon that has energy E_{CMB}. For simplicity, assume that the collision is head-on. At the reaction threshold, what is the energy E_p of the incoming proton in this frame? (Calculate the boost by relating the photon energy E_{CMB} to E_t; you may make an approximation since $\beta_p \approx 1$. Express your answer in terms of the particle masses m_p, m_π, and E_{CMB}.)

(e) [**3 points**] If protons with energies higher than the E_p from part (d) are detected at the Earth, what can you infer about their origin?

Problem I.5

In this problem, we study propagation of plane waves in a homogeneous, nonpermeable ($\mu = 1$), but anisotropic dielectric medium, which is characterized by a symmetric dielectric tensor ϵ_{ij} such that $D_i = \sum_{j=x,y,z} \epsilon_{ij} E_j$.

(a) **[8 points]** Starting from Maxwell's equations (given below), show that a plane wave solution

$$\boldsymbol{E}(\boldsymbol{r}, t) = \tilde{\boldsymbol{E}}(\boldsymbol{k}, \omega)\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{1}$$

with the frequency ω and wave vector $\boldsymbol{k}$ must satisfy the following equation in the Gaussian system

$$(\boldsymbol{k}\cdot\tilde{\boldsymbol{E}})\,\boldsymbol{k} - k^2\tilde{\boldsymbol{E}} + \frac{\omega^2}{c^2}\boldsymbol{\epsilon}\cdot\tilde{\boldsymbol{E}} = 0, \tag{2}$$

where c is the speed of light, and $(\boldsymbol{\epsilon}\cdot\tilde{\boldsymbol{E}})_i \equiv \sum_j \epsilon_{ij}\tilde{E}_j$.

(b) **[9 points]** Suppose x, y, and z are the directions that diagonalize the tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}. \tag{3}$$

Consider a linearly polarized plane wave (1) of the frequency ω traveling in this medium along the direction $\hat{z}$, so that $\boldsymbol{k}\|\hat{z}$. From Eq. (2), find the two possible wave numbers $k_{1,2}$ for this wave and describe their respective polarizations $\tilde{\boldsymbol{E}}_{1,2}$, as well as the corresponding wave lengths $\lambda_{1,2}$.

(c) **[8 points]** Suppose a plane wave of the frequency ω propagates along the direction $\hat{z}$ and is polarized along the direction $\tilde{\boldsymbol{E}}\|(\hat{x}+\hat{y})$ at $z = 0$. At what distance L does the polarization $\tilde{\boldsymbol{E}}$ of the wave turn 90° to become $\tilde{\boldsymbol{E}}\|(\hat{x}-\hat{y})$?

Additional information. For any vector $\boldsymbol{V}$,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{V}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{V}) - \boldsymbol{\nabla}^2\boldsymbol{V}.$$

Maxwell's equations in the absence of free charges and currents are

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0,$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t},$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \frac{1}{c}\frac{\partial\boldsymbol{D}}{\partial t}.$$