# UNIVERSITY OF MARYLAND
## Department of Physics
## College Park, Maryland


# PHYSICS Ph.D. QUALIFYING EXAMINATION
# PART B

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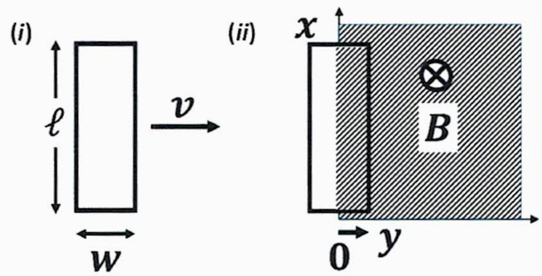
| **January 10, 2024** | **10:00 am – 12:00 pm** |
| **January 11, 2024** | **10:00 am – 12:00 pm** |

**Problem B.1**

Consider a rigid rectangular loop of wire with $\ell \gg w$ shown in Figure (i). The wire has infinite conductivity and circular cross section of radius $a \ll w$.




(a) **[5 points]** Derive an expression for the inductance $L$ of the loop in terms of the length parameters $(\ell, w, a)$ and fundamental constants. Neglect end effects since $\ell \gg w$.

(If you have difficulty solving for $L$, you can proceed to other parts using $L$ as given.)

Initially, the loop does not carry any current and moves with a nonrelativistic speed $v_0$ in the $y$ direction, as shown in Figure (i). Then the loop enters a uniform magnetic field $\boldsymbol{B} = \hat{\boldsymbol{z}} B_0$ at $y > 0$, as shown in Figure (ii). (The plane of the loop remains perpendicular to $\hat{\boldsymbol{z}}$.)

(b) **[5 points]** Calculate the induced current $I(y)$ in the loop as a function of the distance $y$ the right side of the loop has penetrated into the magnetic field, see Figure (ii).

Sketch $I$ vs. $y$ for the three regions: $y < 0$, $0 < y < w$, and $y > w$, and obtain the current $I_*$ for $y > w$.

*Hint: Use the inductance L and the infinite conductivity of the wire.*

(c) **[2 points]** Determine the direction of the induced current in the loop.

(d) **[4 points]** Find the inductive energy $U(y)$ in the loop (due to the current $I(y)$ and inductance $L$) as a function of the distance $y$.

Sketch $U$ vs. $y$ for the three regions: $y < 0$, $0 < y < w$, and $y > w$, and obtain the energy $U_*$ for $y > w$.

(e) **[3 points]** The loop has mass $M$. Utilizing conservation of energy, find the critical initial speed $v_c$ such that for $v_0 > v_c$ the loop keeps moving, whereas for $v_0 < v_c$ it bounces back.

(f) **[3 points]** For $v_0 > v_c$, calculate the terminal speed $v_*$ of the loop after it has fully entered the magnetized region.

(g) **[3 points]** Next consider $v_0 < v_c$ and calculate how much time $\tau$ the right wire segment of the loop spends inside the magnetized region.

Does the answer depend on the initial speed $v_0$?

*Hint: Observe that the kinetic and potential energies of the loop are similar to those of a harmonic oscillator of a certain frequency.*

## Problem B.2

A particle of unknown mass $M$ decays into two particles with the known masses $m_a$ and $m_b$. The goal of this problem is to determine $M$ using observations of the particles $a$ and $b$.

(a) **[6 points]** First, suppose we measure, in the laboratory frame, the energies $E_a$ and $E_b$ of the decay particles and the opening angle $\theta$ between their tracks in the detector.

Express the mass $M$ in terms of $E_a$, $E_b$, $m_a$, $m_b$, $\theta$, and the speed of light $c$.

Next, consider the decay of the charged $W$ boson (which is the carrier of the weak nuclear force) into an electron $e$ of mass $m_e$ and a mass*less* electron anti-neutrino $\nu$. While the electron is detected, the electron anti-neutrino is difficult to observe. In this case, we can*not* use the approach of Part (a) for determining the mass $M_W$ of the $W$ boson. Thus, we need to develop an alternative method, based on the measured energy *distribution* of the visible electron. (Note that the following parts do not make use of the analysis done in Part (a).)

(b) **[6 points]** In the reference frame where the $W$ boson is at *rest*, find the magnitude of the 3-momentum $p_e$ of the electron in terms of $M_W$, $m_e$, and $c$.

(c) **[2 points]** Find the energy $E_e$ of the electron in the same rest-frame of the $W$ boson.

(d) **[2 points]** In the following parts, neglect $m_e$ as compared to $M_W$, since numerically $m_e \approx 0.5$ MeV$/c^2 \ll M_W \approx 80$ GeV$/c^2$.

Write down $E_e$ and $p_e$ in this limit and interpret the result.

(e) **[6 points]** Suppose the $W$ boson is moving in the *laboratory frame* with speed $v$. Let $\alpha$ be the angle between this velocity and the electron velocity in the rest-frame of $W$.

Find the electron energy $E_e^{\text{lab}}$ in the laboratory frame in terms of $M_W$, $v$, $\alpha$, and $c$.

(f) **[2 points]** For a fixed speed $v$ of the $W$ boson, we get a distribution of the electron energy $E_e^{\text{lab}}$ in the laboratory frame as the angle $\alpha$ varies.

Find the maximal $E_{\text{max}}^{\text{lab}}$ and minimal $E_{\text{min}}^{\text{lab}}$ values of the electron energy in terms of $M_W$, $v$, and $c$.

(g) **[1 points]** Find an expression for $M_W c^2$ in terms of $E_{\text{max}}^{\text{lab}}$ and $E_{\text{min}}^{\text{lab}}$.

Show that it is *in*dependent of the speed $v$ of the $W$ boson in the laboratory frame.

## Problem B.3

Examine splitting of the 9-fold degenerate energy level for $n = 3$ of the hydrogen atom due to a weak uniform electric field $\mathcal{E}$ along the $z$ axis. The Hamiltonian of the perturbation is

$$V = -e\mathcal{E}z, \tag{1}$$

where $z$ is the coordinate of the electron relative to the proton.

(a) **[2 points]** Make a list of the states for the principal quantum number $n = 3$ in the notation $|l, m\rangle$, where $l$ is orbital angular momentum and $m$ its projection onto the $z$ axis.

Indicate the parity of these states with respect to the space inversion operation.

(b) **[6 points]** The matrix elements of Eq. (1) in the basis $|l, m\rangle$ form a $9 \times 9$ matrix. Using selection rules for angular momentum and parity, identify all of the nonzero matrix elements.

Show that the matrix consists of decoupled $1 \times 1$, $2 \times 2$, and $3 \times 3$ blocks and identify the states coupled in these blocks.

Denote the nonvanishing matrix elements appearing in the $2 \times 2$ blocks as $A$ and in the $3 \times 3$ block as $B$ and $C$. No need to calculate them explicitly, but note that they are proportional to $\mathcal{E}$, i.e., $A, B, C \propto \mathcal{E}$.

(c) **[3 points]** Obtain the energy eigenvalues and eigenstates for the $1 \times 1$ and $2 \times 2$ blocks (the latter in terms of the matrix element $A$).

Are these energy levels degenerate? Relate your answer to time-reversal symmetry.

(d) **[4 points]** Among the energy eigenstates found in Part (c):

(i) Which are also the eigenstates of the space inversion operation, i.e., have a well-defined parity?

(ii) For which of these states does the parity selection rule allow for a nonzero expectation value $e\langle z\rangle$, the electric dipole moment?

(e) **[6 points]** Obtain the energy eigenvalues for the $3 \times 3$ block in terms of the matrix elements $B$ and $C$. Is there degeneracy among these energy levels?

Show that one eigenenergy is zero and find the corresponding eigenstate (sometimes called the "dark state"). Does it have a well-defined parity and a nonzero $e\langle z\rangle$?

Do the other two energy eigenstates have well-defined parity and nonzero $e\langle z\rangle$?

(f) **[2 points]** Make a sketch of all 9 energy eigenvalues $E$ versus the electric field $\mathcal{E}$, indicating slopes and degeneracies of the energy levels.

(g) **[2 points]** Using the Feynman-Hellman theorem $e\langle z\rangle = -\partial E/\partial\mathcal{E}$, confirm which of the 9 energy eigenstates have nonzero electric dipoles $e\langle z\rangle$.

## Problem B.4

A particle of mass $m$ and energy $E = \hbar^2 k^2/2m$ scatters on a three-dimensional spherically-symmetrical potential of radius $R$:

$$V(r) = \begin{cases} V_0, & r < R, \\ 0, & r > R, \end{cases} \qquad \text{where} \quad V_0 > 0. \tag{1}$$

Consider scattering with zero orbital angular momentum $l = 0$, as represented by the wave function $\psi(r) = u(r)/r$, where the radial function $u(r)$ satisfies a one-dimensional Schrödinger equation $-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + V(r)u(r) = Eu(r)$ for $0 < r < \infty$. Assume that the energy of the particle $E < V_0$ is lower than the height of the potential.

(a) [**4 points**] For $E < V_0$, obtain (up to an overall coefficient) the radial function $u^-(r)$ for $r < R$, satisfying the appropriate boundary condition at $r = 0$.

(b) [**4 points**] The radial function for $r > R$ can be written as $u^+(r) = \sin(kr + \delta_0)$, where $\delta_0$ is the **scattering phase** for $l = 0$.

By matching the boundary conditions for $u^-(r)$ and $u^+(r)$ at $r = R$, obtain a transcendental equation for the scattering phase $\delta_0$.

In the rest of the problem, consider the low-energy limit, where the following conditions are satisfied

$$kR \ll 1 \qquad \text{and} \qquad E \ll V_0, \tag{2}$$

and scattering is dominated by the $l = 0$ channel. Then, the scattering phase $\delta_0$, amplitude $f$, and cross section $\sigma$ can be expressed in terms of a **scattering length** $a_0$ as

$$\delta_0 = -ka_0, \qquad f = -a_0, \qquad \sigma = 4\pi a_0^2. \tag{3}$$

(c) [**5 points**] Taking the low-energy limit (2) in the equation found in Part (b), obtain the scattering length $a_0$ in terms of $R$ and $\kappa_0 = \sqrt{2mV_0/\hbar^2}$. *Hint:* $\tan x \approx x$ for $x \ll 1$.

Sketch a graph of $a_0$ versus the dimensionless parameter $\kappa_0 R$ changing from 0 to $\infty$.

(d) [**4 points**] From the result of Part (c), obtain $a_0$, $f$, and $\sigma$ in the case $V_0 \gg \hbar^2/2mR^2$ corresponding to a strong repulsive potential. Interpret the result.

(e) [**4 points**] Next consider the opposite case of a weak potential: $V_0 \ll \hbar^2/2mR^2$ (while the condition $E \ll V_0$ in Eq. (2) is still satisfied). From the result of Part (c), obtain the scattering length $a_0$ and amplitude $f$ in this case. *Hint:* $\tanh x \approx x - x^3/3$ for $x \ll 1$.

(f) [**4 points**] Calculate the **scattering amplitude** $f$ using the Born approximation in the low-energy limit (2) for a weak potential as in Part (e). Does your result agree with the result for $f$ from Part (e)?

*Useful info:* The amplitude $f(\boldsymbol{k}, \boldsymbol{k}')$ of scattering from $\boldsymbol{k}$ to $\boldsymbol{k}'$ in the Born approximation is

$$f(\boldsymbol{k}, \boldsymbol{k}') = -\frac{m}{2\pi\hbar^2}\int d^3r\, V(\boldsymbol{r})\, e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}. \tag{4}$$