UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART I

January 23, 2020 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

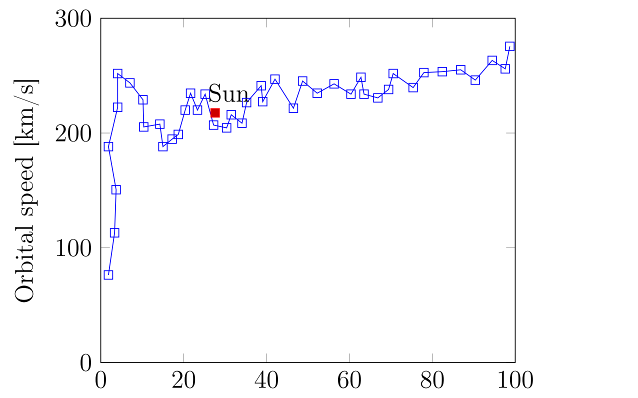
Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1

The rotation curve of a galaxy is a plot of the orbital speeds of visible stars versus their radial distance from the galactic center. This is shown below for our Milky Way galaxy.



(a) **[3 points]** According to Newton/Kepler's law, the orbital speed of a star in the outer parts of the galaxy decreases with distance r from the galactic center. Derive a formula that describes this. You may assume circular orbits, and that the star is sufficiently far out so that the galaxy can be considered to be a point mass M.

(b) **[10 points]** The rotation curve, however, shows a fairly flat speed distribution. Suppose we assume there is a spherically symmetric invisible mass distribution inside and outside the visible galaxy. Let the mass density function (mass per unit volume) be $\rho(r)$, and let the total mass inside radius r be $M(r)$. For a star in a circular orbit inside such a mass distribution, deduce functional forms for $\rho(r)$ and $M(r)$ if the orbital speed is constant independent of r.

Be careful with this part; a significant mistake here will likely lead to significant errors and loss of points in the remaining parts of this question.

(c) **[4 points]** According to direct observations, the density of luminous (visible) matter in our galaxy decreases with distance as $\rho_L(r) \sim r^{-3.5}$. What does this imply about the relative amounts of visible matter and invisible (dark) matter at large r?

(d) **[4 points]** There is evidence, from dwarf galaxies, etc, that the flat speed distribution persists up to 300,000 light years, or, about six times the radius of the Milky Way. If so, what are the implications for the relative mass ratio between visible and dark matter? [Refer to your results in part (b).]

(e) **[4 points]** In one approach, referred to as Modified Newtonian Dynamics, Newton's second law is modified to $F = ma^2/a_0$, for small accelerations $a << a_0$, where a_0 is a constant. Show how this hypothesis could "explain" the constant rotation curve described above. You may assume ordinary kinematic relationships hold as long as you explicitly state any such assumptions you may require.

Problem I.2

A straight wire extends along the z-axis, with $(x_{wire}, y_{wire}) = (0, 0)$. It carries a current $+I$ in the $+z$ direction.

(a) [**5 points**] Write down a vector expression for the magnetic field $\vec{B}$ everywhere. Either cylindrical or cartesian coordinates are acceptable.

(b) [**5 points**] Write down a vector potential such that $\nabla \times \vec{A} = \vec{B}$. Hint: this is best done in cylindrical coordinates.

(c) [**5 points**] Now suppose that the current-carrying wire is slowly moving in the $+y$-direction with a constant velocity v, such that $(x_{wire}, y_{wire}) = (0, vt)$. Obtain an expression for the electric field $\vec{E}$ induced by the moving wire as a function of position and time. Hint: One strategy is to modify the vector potential to accommodate the moving wire coordinates and then differentiate; another is a Lorenz transformation.

(d) [**5 points**] The wire continues to move as in part (c). In addition, a *very thin* plane metal sheet of conductivity σ and permittivity $\mu = \mu_0$ is placed at $x = h$. (The sheet is an Ohmic conductor.) It is parallel to the $y - z$ plane and extends to infinity in both directions. What is the current density $\vec{J}$ in the sheet at $t = 0$ when the moving wire reaches $(x_{wire}, y_{wire}) = (0, 0)$? Neglect any effects of the magnetic fields of the induced currents.

(e) [**5 points**] Considering the sheet to have thickness $\delta x \ll h$ in the x-direction, find an expression for the rate of heat dissipation in a section of the sheet of length L in the z-direction. In your answer, you may leave difficult integrals unevaluated.

Possibly helpful information

In cylindrical coordinates,

$$\nabla \times \vec{V} = \left[\frac{1}{r}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\phi) - \frac{\partial V_r}{\partial \phi}\right]\hat{z} \quad (1)$$

Problem I.3

Treat radiation in a cavity as a low-density gas of ultrarelativistic particles in a container of volume V and at temperature T, where particle momentum $\mathbf{p}$, wavevector $\mathbf{k}$, and energy ε are related by $\varepsilon = |\mathbf{p}|c = \hbar\,|\mathbf{k}|\,c$.

(a) [**5 points**] (i) Derive the partition function $Z_1(T, V)$ of one particle in the gas, writing it to clearly show the dependence on each of the independent variables of the problem as well as the relevant fundamental constants.
You might find helpful the integral $\int_0^\infty e^{-x}\,x^n\,dx = n!$.

(ii) Find $Z(T, V, N)$ for this gas of N particles, saying explicitly whether you are assuming the particles are indistinguishable or distinguishable. (Your choice will not affect your answers in subsequent parts.)

(b) [**5 points**] Show that for this gas the pressure $P = u/3$, where u is the energy density.

(c) [**5 points**] Show that the equation $P\,V^\gamma = \textit{constant}$ is obeyed when one compresses adiabatically the radiation contained in a vessel with perfectly reflecting walls. Determine the value of γ. [Hint: use the First Law in differential form.]

(d) [**5 points**] Derive the Maxwell relation $(\partial S/\partial V)_T = (\partial P/\partial T)_V$. [Hint: use the Helmholtz free energy.]

(e) [**5 points**] Assuming that the energy density of the radiation is completely determined by the temperature T alone (i.e., that $u = u(T)$, with no dependence on N), use the results of parts b) and d) to derive the Stefan-Boltzmann Law: $u(T) = a\,T^4$, where a is a constant.

Problem I.4

Ultra-high-energy cosmic ray protons can lose energy by inelastic collisions with cosmic microwave background (CMB) photons. In this problem, we will look only at the reaction

$$p + \gamma \rightarrow p + \pi^0 \tag{1}$$

Let the proton and pion masses be m_p and m_{π^0}. You may set $c = 1$.

(a) [**10 points**] First consider this reaction in the reference frame in which the proton is initially at rest. What is the minimum 'threshold' energy, E_t, that the photon must have to produce a π^0 by this reaction? Express your answer in terms of m_p and m_{π^0}, respectively.

Hint: At threshold there is no *relative* motion between the final state particles, they act like a single particle of mass $m_p + m_{\pi^0}$.

(b) [**10 points**] Now, consider the same reaction but viewed in the reference frame of the interstellar medium, in which the proton collides with a low energy CMB photon that has energy E_{CMB}. Assume that the collision is head-on. What is the energy E_p of the incoming proton in this frame? Express your answer in terms of the particle masses m_p, m_{π^0}, and E_{CMB}.

(c) [**2 points**] Using your result from part (b), insert the actual masses of the proton and pion ($m_p = 938$ MeV/c^2, $m_{\pi^0} = 135$ MeV/c^2), and a CMB photon energy $E_{CMB} = 10^{-3}$ eV, to calculate E_p in eV.

(d) [**2 points**] Your result in part (c) gives an estimate of the lowest proton energy at which this reaction can occur. In fact, the reaction does not turn on at a single, precise proton energy. Discuss effects that may lead to a softer turn-on.

(e) [**1 point**] If protons with energies much higher than the estimated threshold E_p are detected at the earth what can we conclude about their sources?

Problem I.5

In the limit of small amplitudes, the vertical motion of a horizontal vibrating rod of length L is governed by

$$\frac{\partial^2 y}{\partial t^2} + \Lambda \frac{\partial^4 y}{\partial t^4} = 0 \tag{1}$$

where Λ is a constant that depends on the material and geometrical properties of the rod.

(a) [**5 points**] Confirm that Eq. 1 has valid solutions of the form

$$y(x,t) = cos(\omega t)[B\cos(kx) + C\sin(kx) + D\cosh(kx) + E\sinh(kx)] \tag{2}$$

and find ω in terms of k and Λ.

(b) [**5 points**] The rod is clamped at one end, (say $x = 0$), leading to the boundary conditions $y(0,t) = 0$ and $\partial y(x,t)/\partial x|_{x=0} = 0$. Use these two conditions to eliminate two of the constants in Eq. 2.

(c) [**5 points**] One condition at the free end, $x = L$, is $\partial^3 y(x,t)/\partial x^3|_{x=L} = 0$. Use this to find a relationship between the remaining two constants in Eq. 2.

(d) [**5 points**] Another condition at the free end is $\partial^2 y(x,t)/\partial x^2|_{x=L} = 0$. Show that the allowed values of k satisfy

$$\cos(kL)\cosh(kL) = -1 \tag{3}$$

(e) [**5 points**] Consider Eq. 3 in the limit where kL becomes very large. Show that in this limit $k_n \approx f(n)$, where n is an integer. Determine the function $f(n)$.