UNIVERSITY OF MARYLAND

Department of Physics

College Park, Maryland

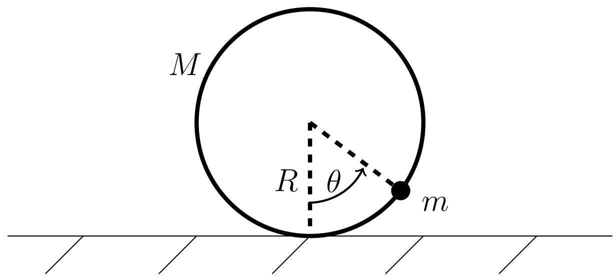
PHYSICS Ph.D. QUALIFYING EXAMINATION

PART A

August 20, 2025	**10:00 am – 12:00 pm**
August 21, 2025	**10:00 am – 12:00 pm**

Problem A.1

A cylindrical hoop of zero thickness, uniformly distributed mass M, and radius R has a point mass m attached to its circumference. The point mass position is characterized by the angle θ measured from the vertical direction as shown in the figure. The hoop rolls without sliding along a horizontal plane under the influence of gravity (the gravitational acceleration is g).



(a) [**5 points**] Show that the kinetic energy of the system is

$$K = MR^2\dot{\theta}^2 + mR^2\dot{\theta}^2(1 - \cos\theta),$$

where dot represents time derivative.

(b) [**5 points**] Obtain the Lagrangian of the system and the equation of motion for the angle $\theta(t)$.

(c) [**5 points**] Keeping terms up to order θ and $\dot{\theta}$ in the equation of motion, determine the frequency ω, and period T, of small oscillations around the stable equilibrium position. Specify the approximation(s) made to arrive at this result.

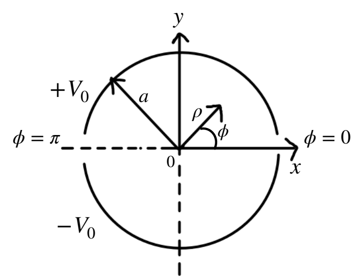
(d) [**5 points**] Return to Part (b) and explicitly set $M = 0$. What is the equation of motion for small oscillations in this case?

Solve this equation exactly and determine time t_0 for the point mass to get from a small initial angle $\theta(0) = \theta_0$ to the bottom $\theta(t_0) = 0$.

(e) [**5 points**] Take the limit $M \to 0$ for the period T in Part (c). Compare the result with the time t_0 in Part (d). Reconcile a discrepancy by examining the assumptions made in each case.

Problem A.2

Consider a thin, hollow, conducting cylinder of radius a, which is infinitely long along the z axis. Its two halves are separated by small lengthwise gaps on each side, and are kept at different potentials $+V_0$ and $-V_0$. The figure below shows the cross section in the (x, y) plane, where the gaps are at $\phi = 0$ and $\phi = \pi$ in cylindrical coordinates (ρ, ϕ, z).



(a) [**1 points**] We are interested in the electric potential $V(\rho, \phi)$, which, obviously, does not depend on z because of translational symmetry.

What is the differential equation satisfied by $V(\rho, \phi)$?

(b) [**4 points**] Using separation of variables in cylindrical coordinates, write the basis functions satisfying the differential equation in Part (a), both inside ($\rho < a$) and outside the cylinder ($\rho > a$), taking into account the expected behavior of $V(\rho, \phi)$ at $\rho \to 0, \infty$ and under the transformation $\phi \to \phi + 2\pi$.

(c) [**5 points**] Write $V(\rho, \phi)$ as a sum over the basis functions from Part (b) with some coefficients C_m using the dimension*less* variable $\tilde{\rho} \equiv \rho/a$. Then determine C_m from the boundary condition at $\rho = a$. (No need to perform the sum over m at this stage.)

For the rest of the problem, consider only inside the cylinder ($\rho < a$).

(d) [**3 points**] By using $V(\rho, \phi)$ from Part (c) for small ρ, find the electric field $\boldsymbol{E}_0 \equiv \boldsymbol{E}(\rho = 0)$ at the origin, both magnitude and direction.

(e) [**4 points**] Calculate the radial component of the electric field $E_\rho(\rho, \phi) = -\partial V(\rho, \phi)/\partial \rho$ by evaluating the sum in Part (c) after taking the derivative. *Hint:* Re-write sines as exponentials, then sum the resulting *geometric* series as $\sum_{n=0}^{\infty} v^n = 1/(1 - v)$.

(f) [**5 points**] Find $V(\rho, \phi)$ by integrating $E_\rho(\rho, \phi)$ from Part (e) over ρ at a constant ϕ, with the boundary condition $V(\rho = 0) = 0$. *Hint:* Use $\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)$.

Check that your final answer for $V(\rho, \phi)$ reproduces $\pm V_0$ at $\rho \to a$.

(g) [**3 points**] Re-write $V(\rho, \phi)$ from Part (f) in Cartesian coordinates as $V(x, y)$. Find and describe an equation for equipotential lines, and sketch them in the (x, y) plane.

The Laplacian in cylindrical coordinates: $\boldsymbol{\nabla}^2 f(\rho, \phi, z) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$

Problem A.3

Consider electrons confined to the two-dimensional (2D) surface (x, y) of a three-dimensional (3D) material. The Hamiltonian of electrons is a 2×2 matrix in the space of spin 1/2

$$H = \frac{\boldsymbol{p}^2}{2m}\,\mathbb{1} + \lambda\,(\hat{\boldsymbol{z}} \times \boldsymbol{p}) \cdot \boldsymbol{\sigma}. \tag{1}$$

The first term is the usual kinetic energy, where $\boldsymbol{p} = (p_x, p_y)$ is the 2D momentum, and $\mathbb{1}$ is the unit matrix in spin space. The second term represents spin-orbit interaction, where $\lambda > 0$ is a coefficient, $\hat{\boldsymbol{z}}$ is a unit vector perpendicular to the plane, and $\boldsymbol{\sigma}$ are the Pauli matrices acting on the electron spin. This term comes from an effective magnetic field along $\hat{\boldsymbol{z}} \times \boldsymbol{p}$, experienced by an electron with momentum $\boldsymbol{p}$, because of the Lorentz transformation of an electric field along $\hat{\boldsymbol{z}}$, originating from termination of the 3D material at the 2D surface.

(a) [**4 points**] Find the eigenvalues $E_\pm(\boldsymbol{p})$ and spinor eigenvectors $|\boldsymbol{p}, \pm\rangle$ of Hamiltonian (1), which are also the eigenstates of momentum $\boldsymbol{p}$. Indicate spin directions for $|\boldsymbol{p}, \pm\rangle$.

(b) [**4 points**] Observe that the eigenenergies $E_\pm$ depend only on the absolute value $p = |\boldsymbol{p}|$. Find the momentum $p_{\min}$ where the energy reaches its minimal value $E_{\min}$ and obtain $E_{\min}$ as well.

(c) [**3 points**] Sketch the two energy branches $E_\pm$ versus p, starting from $p = 0$. Indicate $p_{\min}$ and $E_{\min}$ on your sketch.

(d) [**3 points**] Draw the curves of constant energy $E_\pm(p_x, p_y) = E_F > 0$ for the two branches in the (p_x, p_y) plane. Indicate the directions of spin along these curves.

(e) [**4 points**] Suppose a perpendicular magnetic field B_z is applied along z, so the term $H_z = \mu_B B_z \sigma_z$ is added to Hamiltonian (1). Calculate $E_\pm(p)$ in this case.

(f) [**3 points**] Assuming $\mu_B B_z \ll |E_{\min}|$, show how the sketch of $E_\pm$ versus p is modified compared to Part (c). Find and indicate the energy splitting $E_+ - E_-$ and spin directions at $p = 0$.

(g) [**4 points**] Now suppose that, instead, an in-plane magnetic field B_x is applied along x, so the term $H_x = \mu_B B_x \sigma_x$ is added to Hamiltonian (1).

Calculate $E_\pm(\boldsymbol{p})$ in this case. Find the momentum $\boldsymbol{p}_0$ (both direction and magnitude) where the two branches become degenerate, so that $E_+(\boldsymbol{p}_0) = E_-(\boldsymbol{p}_0) = E_0$, and also find E_0.

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Problem A.4

This problem explores the mean-field theory of a ferromagnetic phase transition. Consider a simple cubic lattice, where each site is occupied by an atom with the spin 1/2 (all atoms are the same). The Hamiltonian for the spins is

$$H = -\mu \sum_i \boldsymbol{B} \cdot \boldsymbol{\sigma}_i - J \sum_{(ij)} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j. \tag{1}$$

The first term, where μ is the magnetic moment of an atom, describes interaction of an external magnetic field $\boldsymbol{B}$ with the spins on sites i represented by the Pauli matrices $\boldsymbol{\sigma}_i$. In the second term, the sum is taken over pairs of neighboring sites (ij) and describes interaction between their spins $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_j$. Assume that the interaction energy $J > 0$ is positive, and the system is at temperature T. Set the Boltzmann constant to $k_B = 1$.

(a) [**4 points**] First consider the case of $J = 0$, i.e., ignore the interaction term in Eq. (1). Calculate the expectation value $\boldsymbol{s} = \langle \boldsymbol{\sigma}_i \rangle$ of the spin on a site i, for a given magnetic field $\boldsymbol{B}$ and temperature T. Specify both the magnitude s and direction of $\boldsymbol{s}$.

(b) [**4 points**] Now treat the interaction J between the spins in mean-field approximation. For a given site i in the second term of Eq. (1), replace $\boldsymbol{\sigma}_j$ by its expectation value $\boldsymbol{s}$ (yet unknown). On a simple cubic lattice, how many neighbors does the site i have?

Then the second term in Eq. (1) produces an additional effective magnetic field acting on the spin $\boldsymbol{\sigma}_i$. Using the result of Part (a), obtain a transcendental equation for s in the presence of B.

(c) [**4 points**] Now set $B = 0$ in the transcendental equation for s obtained in Part (b). Show graphically that the resulting equation has only a trivial solution $s = 0$ for $T > T_c$, but acquires a nontrivial solution $s \neq 0$ for $T < T_c$. Determine the ferromagnetic transition temperature T_c (the Curie temperature) in terms of J.

(d) [**4 points**] From the transcendental equation in Part (c) for $B = 0$, find the (nonzero) value of s in the limit $T \to 0$. Interpret the result.

(e) [**5 points**] From the transcendental equation in Part (c) for $B = 0$, find $s(T)$ for $T < T_c$ in the vicinity of the transition temperature $T_c - T \ll T_c$. Express the answer in terms of T and T_c, and discuss the limit of $s(T)$ at $T \to T_c^-$.

Hint: Assume that s is small and use Taylor expansion in s (see the info at the bottom).

(f) [**4 points**] Now go back to the transcendental equation for $B \neq 0$ obtained in Part (b). From this equation, obtain the spin susceptibility $\chi(T) = ds/dB$ at $B = 0$ for $T > T_c$. Express the answer in terms of $T - T_c$ and discuss the behavior of $\chi(T)$ at $T \to T_c^+$.

Useful information: $\tanh x \approx x - x^3/3$ for $x \ll 1$.