UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART A
August 23, 2021 10:00 am – 12:00 pm

Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet – not your name!

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.
Problem A.1

A surprisingly accurate approximation for the motion of a mass \( m \) orbiting a black hole of mass \( M \) can be obtained by using ordinary nonrelativistic Newtonian mechanics, but slightly modifying the usual \( 1/r \) potential to

\[
U(r) = -\frac{GmM}{(r - r_g)}.
\]

(1)

Here \( G \) is the gravitational constant, and \( r_g \) is the radius of the black-hole event horizon. Orbits with \( r < r_g \) are inside the black hole and, so, unphysical. Assume that \( M \gg m \), so the black hole is stationary.

(a) [6 points] By means of a Lagrangian or otherwise, obtain, for a general potential \( U(r) \) and orbital angular momentum \( \ell \), an equation for the radius \( r(t) \) in the form

\[
m \frac{d^2r}{dt^2} + \frac{\partial}{\partial r} W_{\text{eff}}(r, \ell) = 0,
\]

(2)

with an explicit formula for the effective radial potential \( W_{\text{eff}}(r, \ell) \).

(b) [6 points] Using the potential \( W_{\text{eff}}(r, \ell) \) obtained above, find the value of \( \ell \) that allows a circular orbit of a given radius \( r_0 \) around the black hole.

(c) [6 points] Explain how you would use some property of \( W_{\text{eff}}(r, \ell) \) to determine whether the circular orbit found in Part (b) is stable or unstable, when the particle is given a small kick that does not alter its orbital angular momentum \( \ell \).

(d) [7 points] On the basis of Part (c), find the critical radius \( r_c \) separating stable circular orbits for \( r_0 > r_c \) from unstable orbits in the range \( r_g < r_0 < r_c \) for the potential \( U(r) \).
Problem A.2

Consider a very long transmission line of length $\ell$ consisting of two perfectly conducting wires running parallel to the $x$ axis at $y = 0$ and $y = a > 0$ in vacuum, as shown in the figure. The wires have capacitance $\tilde{C} = C/\ell$ and inductance $\tilde{L} = L/\ell$ per unit length. This problem concerns electromagnetic waves in this system with wavelengths $\lambda$ much longer than the distance $a$ between the wires: $\lambda \gg a$, but much shorter than the wires length: $\lambda \ll \ell$.

\[ I(x,t) \]
\[ V(x,t) \]
\[ I(x,t) \]

(a) [5 points] Let $V(x,t)$ be voltage between the wires, defined as the integral of the electric field along a straight path between the wires at fixed $x$ and $t$

\[ V(x,t) = - \int_0^a E_y(x,y,t) \, dy. \]  

Show that

\[ \frac{\partial V(x,t)}{\partial x} = \kappa \frac{\partial I(x,t)}{\partial t} \]  

with some constant $\kappa$ depending on $\tilde{L}$ and/or $\tilde{C}$ that you should find. Here $I(x,t)$ is the current at $x$ in the lower ($y = 0$) wire, whereas the current in the upper ($y = a$) wire has the same magnitude, but opposite direction.

*Hint:* The magnetic flux through a rectangular area bounded by the wires and a very small horizontal width $\Delta x$ is $\Delta \Phi_B = I(x,t) \tilde{L} \Delta x$. Apply Faraday's law to this area, taking into account that the electric field is zero inside an ideal wire of zero resistance.

(b) [5 points] By using the charge continuity equation, derive a similar relation between $\partial V(x,t)/\partial t$ and $\partial I(x,t)/\partial x$.

*Hint:* The electric charges induced in short segments of width $\Delta x$ at the top and bottom wires are related to capacitance as $\Delta Q = \pm V(x,t) \tilde{C} \Delta x$.

(c) [5 points] Combining the differential equations from Parts (a) and (b), obtain a wave equation for $I(x,t)$ and express the wave velocity $v$ in terms of $\tilde{L}$ and $\tilde{C}$.

(d) [5 points] For the wave $I(x,t) = I_0 \sin[k(x - vt)]$ propagating to the right, find the corresponding $V(x,t)$.

Suppose $I_0 > 0$, and $\sin[k(x - vt)] = 1$ for given values of $x$ and $t$. According to Eq. (1), what is the direction of $E_y(x,y,t)$ for the same $x$ and $t$: up or down in the figure?

(e) [5 points] Suppose the right end of the transmission line is terminated by a resistor $R$ connecting the two wires at $x = \ell$. Find the value of $R$ such that the wave $I(x,t) = I_0 \sin[k(x - vt)]$ propagating to the right is completely absorbed at $x = \ell$ and does not generate a reflected wave.