Start each problem on a new sheet of paper.

Be sure to write your Qualifier ID ("control number") at the top of each sheet – not your name!

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.
Problem A.3

Consider a particle of mass $m$ in an asymmetric one-dimensional potential of width $a$ and depth $V_0 > 0$:

$$V(x) = \begin{cases} 
\infty, & -\infty < x < 0, \\
-V_0, & 0 < x < a, \\
0, & a < x < \infty.
\end{cases} \quad (1)$$

(a) [9 points] Derive a transcendental equation that determines the energies $E$ of bound states for this potential.

(b) [4 points] What is the minimum depth $V_0$ for which a bound state exists?

(c) [4 points] How many bound states are there for a general depth $V_0$?

(d) [4 points] Suppose the system has a shallow bound state with the energy $E = -0.01 \frac{\hbar^2}{2ma^2}$. Estimate the probability $P$ to find the particle inside the potential well, with the coordinate $0 < x < a$.

(e) [4 points] Suppose the potential (1) has bound states. Now let us modify the potential by adding a positive part outside of the negative part:

$$V(x) = \begin{cases} 
\infty, & -\infty < x < 0, \\
-V_0, & 0 < x < a, \\
+W_0, & a < x < b, \\
0, & b < x < \infty,
\end{cases} \quad (2)$$

where $W_0 > 0$.

Describe qualitatively how the presence of the positive part modifies energies of the bound states compared with the case $W_0 = 0$. Are they lowered, raised, or unchanged?
Problem A.4

This problem deals with the first quantum mechanical model that can account for the observation that the heat capacity per volume $C_V(T)$ of insulators is much smaller at low temperatures than the classical result (the Dulong-Petit Law).

(a) [4 points] Derive an expression for the average energy at temperature $T$ of a quantum harmonic oscillator of the angular frequency $\omega$ in one dimension $(D = 1)$.

(b) [4 points] Mindful of Planck’s results and the quantum theory of oscillators, Albert Einstein proposed a crude model of insulators. He set all the vibrational modes of the $N$ atoms in a three-dimensional $(D = 3)$ solid insulator of volume $V$ to have the same frequency $\omega_E$. Find $C_V(T)$ of this so-called Einstein model.

(c) [2 points] How would $C_V(T)$ change if the problem were formulated in one- or two-dimensional space?

(d) [3 points] Find the high-temperature limit of $C_V$ in $D = 3$ and verify that it agrees with the classical result coming from the Equipartition Theorem.

(e) [3 points] Find the expression to which $C_V(T)$ simplifies at temperatures well below $\hbar \omega_E/k_B$, and then evaluate $C_V(0)$.

(f) [2 points] Sketch the behavior of $C_V(T)$ vs. $T$ from $T = 0$ to a temperature a few times $\hbar \omega_E/k_B$.

(g) [2 points] Phonon modes in a solid can be acoustic or optical. Which of these modes are better described by the Einstein model?

(h) [3 points] Why does the Einstein model poorly describe the magnitude and thermal behavior of $C_V(T)$ of a metal?

(i) [2 points] Within the Einstein model, compare two single-element insulators, both having the same $N$ and $V$ but with each made entirely of one of two different isotopes of the element. How, if at all, would $C_V(T)$ differ? In the limits of high and of low temperature, would $C_V$ be larger or smaller for the insulator with the higher-mass isotope?