UNIVERSITY OF MARYLAND

Department of Physics

College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION

PART I

August 23, 2021 9:00 am – 1:00 pm

Do any four problems. Each problem is worth 25 points. Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet – not your name! – and turn in solutions to four problems only. (If five solutions are turned in, we will only grade #1 - #4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1

A surprisingly accurate approximation for the motion of a mass m orbiting a black hole of mass M can be obtained by using ordinary nonrelativistic Newtonian mechanics, but slightly modifying the usual $1/r$ potential to

$$U(r) = -\frac{GmM}{(r - r_g)}. \tag{1}$$

Here G is the gravitational constant, and r_g is the radius of the black-hole event horizon. Orbits with $r < r_g$ are inside the black hole and, so, unphysical. Assume that $M \gg m$, so the black hole is stationary.

(a) [**6 points**] By means of a Lagrangian or otherwise, obtain, for a general potential $U(r)$ and orbital angular momentum ℓ, an equation for the radius $r(t)$ in the form

$$m\frac{d^2r}{dt^2} + \frac{\partial}{\partial r}W_{\text{eff}}(r, \ell) = 0, \tag{2}$$

with an explicit formula for the effective radial potential $W_{\text{eff}}(r, \ell)$.

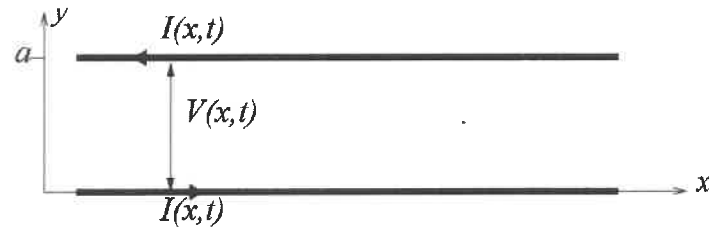
(b) [**6 points**] Using the potential $W_{\text{eff}}(r, \ell)$ obtained above, find the value of ℓ that allows a circular orbit of a given radius r_0 around the black hole.

(c) [**6 points**] Explain how you would use some property of $W_{\text{eff}}(r, \ell)$ to determine whether the circular orbit found in Part (b) is stable or unstable, when the particle is given a small kick that does not alter its orbital angular momentum ℓ.

(d) [**7 points**] On the basis of Part (c), find the critical radius r_c separating stable circular orbits for $r_0 > r_c$ from unstable orbits in the range $r_g < r_0 < r_c$ for the potential $U(r)$.

Problem I.2

Consider a very long transmission line of length ℓ consisting of two perfectly conducting wires running parallel to the x axis at $y = 0$ and $y = a > 0$ in vacuum, as shown in the figure. The wires have capacitance $\tilde{C} = C/\ell$ and inductance $\tilde{L} = L/\ell$ per unit length. This problem concerns electromagnetic waves in this system with wavelengths λ much longer than the distance a between the wires: $\lambda \gg a$, but much shorter than the wires length: $\lambda \ll \ell$.



(a) **[5 points]** Let $V(x,t)$ be voltage between the wires, defined as the integral of the electric field along a straight path between the wires at fixed x and t

$$V(x,t) = -\int_0^a E_y(x,y,t)\,dy. \tag{1}$$

Show that

$$\frac{\partial V(x,t)}{\partial x} = \kappa\,\frac{\partial I(x,t)}{\partial t} \tag{2}$$

with some constant κ depending on $\tilde{L}$ and/or $\tilde{C}$ that you should find. Here $I(x,t)$ is the current at x in the lower ($y = 0$) wire, whereas the current in the upper ($y = a$) wire has the same magnitude, but opposite direction.

Hint: The magnetic flux through a rectangular area bounded by the wires and a very small horizontal width Δx is $\Delta\Phi_B = I(x,t)\,\tilde{L}\,\Delta x$. Apply Faraday's law to this area, taking into account that the electric field is zero inside an ideal wire of zero resistance.

(b) **[5 points]** By using the charge continuity equation, derive a similar relation between $\partial V(x,t)/\partial t$ and $\partial I(x,t)/\partial x$.

Hint: The electric charges induced in short segments of width Δx at the top and bottom wires are related to capacitance as $\Delta Q = \pm V(x,t)\,\tilde{C}\,\Delta x$.

(c) **[5 points]** Combining the differential equations from Parts (a) and (b), obtain a wave equation for $I(x,t)$ and express the wave velocity v in terms of $\tilde{L}$ and $\tilde{C}$.

(d) **[5 points]** For the wave $I(x,t) = I_0 \sin[k(x - vt)]$ propagating to the right, find the corresponding $V(x,t)$.

Suppose $I_0 > 0$, and $\sin[k(x - vt)] = 1$ for given values of x and t. According to Eq. (1), what is the direction of $E_y(x,y,t)$ for the same x and t: up or down in the figure?

(e) **[5 points]** Suppose the right end of the transmission line is terminated by a resistor R connecting the two wires at $x = \ell$. Find the value of R such that the wave $I(x,t) = I_0 \sin[k(x - vt)]$ propagating to the right is completely absorbed at $x = \ell$ and does not generate a reflected wave.

Problem I.3

Consider a binary alloy where each site of a lattice is occupied by an atom of type A or B. Each lattice site has c nearest neighbors, and only nearest neighbor (NN) atoms interact. When two A atoms are on NN sites, their interaction energy is $-\varepsilon_{AA}$, and similarly $-\varepsilon_{BB}$ for two B atoms and $-\varepsilon_{AB}$ for A and B atoms, where the parameters $\varepsilon > 0$ are all positive. Let the numbers of the two kinds of atoms in a sample of the alloy be N_A and N_B, with $N_A + N_B = N$. The concentrations are $n_A = N_A/N$ and $n_B = N_B/N$, and the difference of concentrations is $x = n_B - n_A$.

(a) **[5 points]** At high enough temperature T, the atoms are *randomly distributed* among the lattice sites, one atom per site. Find the average energy U per site for such a configuration. First obtain U in terms of n_A and n_B, and then write $U(x)$ as a function of x. Finally, write $U(x)$ once more, with the simplifying assumption $\varepsilon_{AA} = \varepsilon_{BB} \equiv \varepsilon_0$.

(b) **[1 points]** Assuming $\varepsilon_{AB} < \varepsilon_0$ for the rest of the problem, sketch a plot of the function $U(x)$, indicating the relevant range of x and the locations of the extrema of $U(x)$. Is $U(x)$ lower at $x = 0$ or at $|x| = 1$?

(c) **[5 points]** (i) Calculate the configurational entropy S per site for randomly distributed atoms, assuming $N_A, N_B \gg 1$ so that the Stirling approximation, $\ln(N!) \approx N \ln N - N$, can be used. First find S in terms of n_A and n_B, then express it as $S(x)$.

(ii) Sketch a plot of $S(x)$ indicating clearly the value of x for which $S(x)$ is maximal, and the values $S(\pm 1)$.

(d) **[5 points]** Using the previous results, obtain the free energy per site, $F(x,T)$, and sketch plots of F vs. x at three temperatures, $T_1 > T_2 > T_3$, chosen so that $F(x)$ has one minimum at T_1, two minima and one local maximum at T_3 (not counting extrema at $x = \pm 1$), and T_2 is the temperature on the border between the two behaviors.

(e) **[5 points]** At a fixed temperature T, a binary alloy with a given x may stay in the randomly distributed uniform phase, or it may spontaneously segregate into regions, each having one of two different (temperature-dependent) values of x. A local segregation instability exists when $\partial^2 F(x,T)/\partial x^2 < 0$ (because then a small amount of segregation can lower the free energy).

(i) Determine T_*, the highest T at which the local segregation instability can occur, and the value of x for which this happens.

(ii) For a given $T < T_*$, find the range of locally unstable x values and indicate this range on the sketch of $F(x,T)$ versus x.

(f) **[4 points]** Sketch a very rough phase diagram of the binary alloy model, with $|x| \leq 1$ on the horizontal axis and T on the vertical axis. Draw and label a curve marking the boundary of the region unstable to local segregation, and another curve marking the x values of the free energy minima at each T. Label the regions bounded by these curves, and describe the nature of the phase in each region.

Problem I.4

In galaxies like our own, the visible stars and gas appear to be embedded in a much more massive cloud of invisible "dark matter." If the dark matter consists of χ particles of mass m_χ, and if these particles possess non-gravitational interaction of some kind with the ordinary matter, we might detect them in several ways.

(a) [**9 points**] *Accelerator Production.* We may try to produce dark-matter particles in an electron-positron collider via the reaction $e^+ + e^- \to \chi + \chi$. For a head-on collision of electrons and positrons with the unequal total relativistic energies E_- and E_+ in the laboratory frame, what is the largest mass m_χ of the χ particle that this collider can produce? Assume that E_- and E_+ are large enough that we can neglect the masses of the electrons and positrons.

(b) [**9 points**] *Recoil Detection.* The dark-matter χ particles traveling at the typical galactic orbital speeds should be constantly incident on nuclei in a terrestrial detector. Recoil of a nucleus from elastic scattering of a χ particle may be detectable. Suppose that a χ particle with the lab-frame speed $10^{-3}c$ strikes a stationary Ge nucleus of mass $m_G \approx 70$ GeV$/c^2$. If our detector can only identify nuclear recoil with kinetic energy greater than $E_G = 10$ keV in the lab frame, what is the smallest value of m_χ (in units of GeV$/c^2$) that can be detected?

For a numerical answer, approximate a square root the best you can.

(c) [**7 points**] *Gamma Background.* A possible source of false-positive background events in the recoil experiment of Part (b) is gamma radiation arising from radioactive decay of surrounding materials. What is the minimum energy E_γ that a photon must have in order to impart the recoil kinetic energy $E_G = 10$ keV to a Ge nucleus by scattering (not absorption)?

Given that the energy of gamma rays from long-lived radioisotopes never exceeds 4 MeV, would this background be a problem for the experiment in Part (b)?

For a numerical answer, approximate a square root the best you can.

Problem I.5

An infinitely long string with linear (1D) mass density μ, stretched to tension τ, runs along the x axis in its equilibrium configuration, and vibrates with small amplitude in the x-y plane.

(a) [**3 points**] Write the linear wave equation satisfied by the transverse displacement $y(x,t)$, and indicate how the wave speed c depends on the parameters of the system. (A derivation is not required.)

(b) [**3 points**] Write the most general solution $y(x,t)$ for a displacement of the string with frequency ω, including all necessary arbitrary constants of integration.

(c) [**7 points**] Now suppose a point mass m is attached at $x = 0$. What are the conditions relating the displacements $y_-(x,t)$ $(x < 0)$ and $y_+(x,t)$ $(x > 0)$ and their derivatives on either side of the point mass?

(d) [**7 points**] Find the reflection and transmission amplitudes R and T for a wave of frequency ω that is initially incoming from the negative x direction. Treat the wave as a complex function whose real part represents the physical wave. Check that your result has the correct value in the limits $m = 0$ and $m = \infty$.

(e) [**5 points**] State and verify the property of R and T that expresses conservation of the wave energy.