# UNIVERSITY OF MARYLAND

## Department of Physics

## College Park, Maryland

# PHYSICS Ph.D. QUALIFYING EXAMINATION

## PART A

**August 16, 2023** **10:00 am – 12:00 pm**

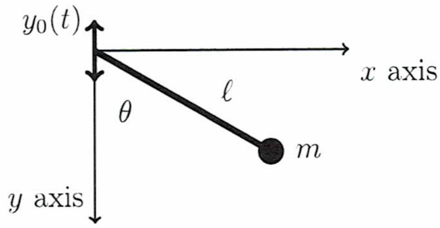
**Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).**

**Be sure to write your Qualifier ID ("control number") at the top of each sheet – not your name!**

**You may keep this packet with the questions after the exam.**

## Problem A.1

A massless rod of length $\ell$ with a point mass $m$ at one end is suspended at the other end. The axes $x$ and $y$ are shown in the figure, where gravity (with acceleration $g$) acts downward along positive $y$. The point of suspension is externally driven vertically as $y_o(t) = a\cos(\gamma t)$ in time $t$ with frequency $\gamma$ around the origin $x = y = 0$. Assume that the drive amplitude is small $a \ll \ell$, and the drive frequency $\gamma$ is much greater than the natural oscillation frequency of the pendulum.




(a) **[4 points]** Express the kinetic and potential energies of the pendulum in terms of the angle $\theta$ between the rod and the $y$ axis. (To save space, retain the general notation $y_o(t)$ until further notice.)

Write down the Lagrangian $\mathcal{L}(\theta, \dot{\theta}, t)$ for the system. Omit additive terms that do not depend on $\theta$, because they do not affect the equations of motion.

(b) **[3 points]** Write down Lagrange's equation of motion. For the case of a stationary point of suspension ($a = 0$), find the frequency $\omega_0$ of small oscillations around $\theta = 0$.

(c) **[4 points]** Introduce generalized momentum $P_\theta = \partial\mathcal{L}/\partial\dot{\theta}$, which is the angular momentum. Then obtain the Hamiltonian $H(\theta, P_\theta, t)$ of the system.

(d) **[3 points]** Write down Hamilton's equations of motion for the system. Verify that they are in agreement with Lagrange's equations obtained earlier in Part (b).

(e) **[4 points]** Obtain the effective Hamiltonian $H_{\text{eff}}(\theta, P_\theta) = \overline{H(\theta, P_\theta, t)}$ by time-averaging over the fast oscillating $y_0(t)$. (*Hint:* $\overline{\ddot{y}_0(t)} = 0$, but $\overline{\dot{y}_0^2(t)} \neq 0$.) Use the explicit form $y_o(t) = a\cos(\gamma t)$ here and write your answers in terms of $a$ and $\gamma$.

Show that $H_{\text{eff}}(\theta, P_\theta) = P_\theta^2/2m\ell^2 + U_{\text{eff}}(\theta)$ and find the effective potential $U_{\text{eff}}(\theta)$.

(f) **[3 points]** Sketch $U_{\text{eff}}(\theta)$ for (i) a small $\gamma$ (but still $\gamma \gg \omega_0$) and (ii) a big enough $\gamma$, and identify minima of the effective potential in these two cases.

Show that $\theta = 0$, i.e., the pendulum pointing downward, is always a minimum of $U_{\text{eff}}$, and therefore an equilibrium position.

Obtain the frequency $\omega_1$ of small oscillations around this minimum.

(g) **[4 points]** Derive a condition on the drive frequency $\gamma$ involving $a$, $g$, and $\ell$ for which the pendulum also has a stable equilibrium at $\theta = \pi$, i.e., pointing directly *upward*.

Obtain the frequency $\omega_2$ of small oscillations around that equilibrium point.

## Problem A.2

Here we estimate the rate of thermonuclear reactions in collisions between nuclei in a star.

(a) **[3 points]** Consider a collision between two nuclei, deuterium and tritium, with masses $m_1$ and $m_2$, and electric charges $e$. We focus on their *relative* motion in the center-of-mass frame, characterized by relative velocity $\boldsymbol{v}$ and the associated momentum $\boldsymbol{p} \equiv \mu \boldsymbol{v}$, where $\mu \equiv m_1 m_2/(m_1 + m_2)$ is the *reduced* mass. The differential of the relative flux $\Phi$ (per unit volume) of one nucleus toward the other can be expressed as

$$d\Phi = C\, v\, e^{-E(p)/T}\, d^3p, \tag{1}$$

where $C$ is a given dimensional constant, which depends on parameters other than the variable $\boldsymbol{p}$. The exponential factor represents the thermal Maxwell-Boltzmann probability density for the energy $E(p) = p^2/2\mu$ of relative motion. The Boltzmann constant is set to 1, and temperature $T$ is measured in energy units.

Express $d\Phi$ in Eq. (1) in terms of $E$ and $dE$.

(b) **[6 points]** For fusion reaction to happen, a nucleus with the relative energy $E$ and reduced mass $\mu$ has to tunnel into the other nucleus through a Coulomb barrier. In the WKB approximation, the cross section $\sigma(E)$ of this reaction can be expressed as

$$\sigma(E) = \sigma_0 e^{-2\gamma(E)}, \qquad \gamma(E) = \frac{1}{\hbar}\int_0^{r_*(E)} \sqrt{2\mu\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{r} - E\right)}\, dr = \frac{b}{\sqrt{E}}, \tag{2}$$

where $\sigma_0$ is a given constant, and $r_*(E) = e^2/4\pi\epsilon_0 E$ is the classical turning point.

Evaluate the integral in Eq. (2) and calculate the factor $b$. *Hint:* Use either a trigonometric substitution, or $z = \sqrt{r}$ and contour integration in the complex plane of $z$.

(c) **[3 points]** Combining Eqs. (1) and (2), the rate $\Gamma$ of nuclear reactions (per unit volume) can be expressed as

$$\Gamma = \int \sigma(E)\, d\Phi = \int_0^\infty G(E)\, e^{-F(E)}\, dE. \tag{3}$$

Identify the functions $F(E)$ and $G(E)$.

(d) **[5 points]** Calculate the integral in Eq. (3) using the saddle point approximation (the method of steepest descent) as follows. First, find the value $E = E_m$ corresponding to the *minimum* of the function $F(E)$ and evaluate $F(E_m)$ and $G(E_m)$.

(e) **[5 points]** Then, expand the function $F(E)$ around its minimum and evaluate the resulting Gaussian integral in Eq. (3). Assume $T \ll b^2$ and use $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$.

(f) **[3 points]** Estimate $b^2/T$ and $F(E_m)$ in the Sun's core where $T \approx 15\times 10^6$ K $\approx 1.3$ keV. *Hint:* The Rydberg energy is $(m_e/2\hbar^2)(e^2/4\pi\epsilon_0)^2 = 13.6$ eV; the mass ratio of nucleon (proton or neutron) and electron is $m_n/m_e \approx 2\times 10^3$; take $m_1 \approx 2m_n$ and $m_2 \approx 3m_n$.

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## Problem A.3

A density matrix operator $\hat{\rho}$ in a quantum Hilbert space has the following properties: $\hat{\rho}$ is Hermitian, $\text{Tr}[\hat{\rho}] = 1$, and all eigenvalues of $\hat{\rho}$ are greater than or equal to zero. In a diagonal basis, it has the following form:

$$\hat{\rho} = \sum_j p_j |\phi_j\rangle\langle\phi_j| \quad \text{with} \quad \sum_j p_j = 1, \quad p_j \geq 0 \text{ for all } j, \quad \langle\phi_j|\phi_k\rangle = \delta_{jk}. \tag{1}$$

A system described by $\hat{\rho}$ has (classical) probability $p_j$ to be in quantum state $|\phi_j\rangle$. The system is in a **pure state** if $p_j = 1$ for a single $j$; otherwise it is in a **mixed state**.

(a) **[3 points]** Show that the expectation value of an operator $\hat{A}$ is $\langle\hat{A}\rangle = \text{Tr}[\hat{\rho}\hat{A}]$.

(b) **[3 points]** Suppose that $\hat{\rho}$ satisfies $\hat{\rho}^2 = \hat{\rho}$. Does it represents a pure or mixed state?

(c) **[3 points]** Show in general that $\text{Tr}[\hat{\rho}^2] \leq \text{Tr}[\hat{\rho}] = 1$. What happens for a pure state?

(d) **[3 points]** The von Neumann entropy is defined as $S = -\text{Tr}[\hat{\rho}\ln\hat{\rho}]$, where the operator $\ln\hat{\rho}$ is $\sum_j \ln(p_j)|\phi_j\rangle\langle\phi_j|$ in the diagonal basis.

Express $S$ in terms of $p_j$. What is the value of $S$ in a pure state?

For the rest of the problem, consider a two-state system represented by **spin 1/2**.

(e) **[3 points]** Show that the most general form of a density matrix in this case is

$$\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}), \tag{2}$$

where $\boldsymbol{n}$ is a three-component vector, and $\hat{\boldsymbol{\sigma}}$ is a vector of the three Pauli matrices (shown at the bottom) in the basis of the up $|\uparrow\rangle$ and down $|\downarrow\rangle$ spin states.

What is a permissible range for the magnitude $n = |\boldsymbol{n}|$? What is $n$ for a pure state?

(f) **[3 points]** What is the expectation value of the spin operator vector $\langle\hat{\boldsymbol{\sigma}}\hbar/2\rangle$ in state (2)? Specify its direction and magnitude, and compare with the case of a pure state.

(g) **[3 points]** What is the von Neumann entropy $S$ in state (2)?

What are the minimal and maximal possible values of $S$?
For which values of $n$ are they achieved?

(h) **[4 points]** Consider $\hat{\rho} = \frac{3}{4}|\uparrow\rangle\langle\uparrow| + \frac{i}{4}|\downarrow\rangle\langle\uparrow| - \frac{i}{4}|\uparrow\rangle\langle\downarrow| + \frac{1}{4}|\downarrow\rangle\langle\downarrow|$. For this state:

(i) What is the vector $\boldsymbol{n}$ and its magnitude $n$ in Eq. (2)?

(ii) What is the expectation value of the spin operator vector $\langle\hat{\boldsymbol{\sigma}}\hbar/2\rangle$?

(iii) What is the von Neumann entropy $S$? (No need to evaluate it numerically.)

(iv) Is this state pure or mixed?

Pauli matrices: $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

## Problem A.4

Consider a gas of particles of mass $m$ moving freely inside a container of volume $V$. The particles can also occupy bound states of negative energy $\epsilon_0 < 0$ located at sites on the surface of the container. There are $N_{\text{sites}}$ such sites, and each site can be either empty or occupied by one particle at most. We use the convention where zero energy corresponds to a free particle with zero kinetic energy.

(a) **[4 points]** For given temperature $T$ and chemical potential $\mu$, calculate the average number of occupied surface states $N_{\text{surf}}$.

Does it matter for your answer whether the particles are bosons or fermions?

(b) **[4 points]** In the rest of the problem, assume that the **particles are bosons**. For the same $T$ and $\mu$, derive a formula for the number of particles $N_{\text{vol}}$ inside the volume of the container (not including the surface sites). Express your answer as an integral, but do not try to evaluate this integral.

In this Part, assume that the system is at a temperature above Bose-Einstein condensation (to be discussed below). Explain what is the permissible range for the chemical potential $\mu$ in this case.

(c) **[2 points]** Suppose the total number of particles in the container is $N$, shared between volume and surface states. Combining the results of Part (a) and (b), obtain an implicit equation for the chemical potential $\mu$ for given $N$ and $T$. (Do not try to solve it.)

(d) **[4 points]** Using the result of Part (c), obtain an implicit equation for the temperature $T_c$ of Bose-Einstein condensation for a given $N$.

What is the value of $\mu$ at $T_c$?

Does the presence of surface states increase or decrease $T_c$, relative to the case of the same $N$ but without surface sites?

(e) **[4 points]** What is the value of $\mu$ at $T < T_c$?

Obtain a formula for the number $N_{\text{cond}}(T)$ of particles in the Bose condensate at $T < T_c$.

What is the value of $N_{\text{cond}}(0)$ at $T = 0$, assuming $N > N_{\text{sites}}$?

(f) **[3 points]** Is Bose-Einstein condensation possible at $T = 0$ in the case $N < N_{\text{sites}}$? Interpret your answer qualitatively.

(g) **[4 points]** Assuming that $N > N_{\text{sites}}$, consider a sufficiently low temperature such that $T \ll T_c$ and $T \ll |\epsilon_0|/k_B$, where $k_B$ is the Boltzmann constant.

In this limit, show that the heat capacity $C(T)$ has a power-law dependence on temperature $C \sim T^\alpha$ and obtain the exponent $\alpha$.