UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

August 26, 2016 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

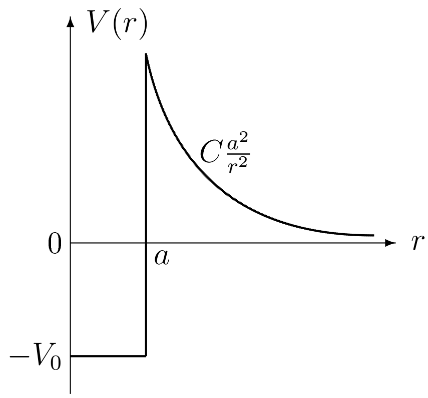
You may keep this packet with the questions after the exam.

Problem II.1

A particle of mass m moves in a 3-dimensional potential

$$V(r) = \begin{cases} -V_0, & 0 < r < a \\ Ca^2/r^2, & r > a \end{cases}$$

where r is the distance of the particle from the origin and the three constants C, a, and V_0 are positive.



(a) [**7 points**] Show that the zero-energy s-wave solutions of Schrödinger's equation in the region $r > a$ are of the form r^ν and $r^{-\nu-1}$, where ν is a positive real number.

(b) [**6 points**] Determine ν in terms of C, m, a, and $\hbar$. What is the appropriate radial dependence of the wavefunction for $r > a$ for a bound state of infinitesimally small binding energy?

(c) [**6 points**] Find a condition on V_0 in terms of C, m, a, and $\hbar$ such that there is exactly one bound s-wave state of infinitesimally small binding energy.
Hint: to simplify the algebra, define the rms momentum of the particle inside the well, $\hbar k = \sqrt{2mV_0}$ and write the condition in terms of k.

(d) [**6 points**] V_0 happens to be such that the rms momentum of the particle inside the well is $3\pi\hbar/4a$. Find the numerical value of V_0/C for this special case.

Problem II.2

An electron of mass m moves in a one-dimensional attractive potential $U(x) = -\lambda\delta(x)$, where $\delta(x)$ is the Dirac delta function and $\lambda > 0$.

(a) **[5 points]** Find the wave function and the energy E_0 of the bound state. What is the parity of the wave function with respect to the operation $x \to -x$?

(b) **[5 points]** Find the wave functions and the energies of the unbound states which are antisymmetric with respect to the parity operation $x \to -x$. Because they are not square-integrable, normalize them such that total $|\psi|^2$ in one wavelength is unity.

For time $t < 0$, the electron is in the ground state of the potential. At time $t = 0$, a small AC electric field $\mathcal{E}(t) = \mathcal{E}_0 \sin(\omega t)$ with frequency $\omega > |E_0|/\hbar$ is turned on. The Hamiltonian of the perturbation is

$$V = -2ex\mathcal{E}_0 \sin(\omega t)$$

where e is the electron charge. The perturbation may cause a transition from the bound state to one of the unbound states.

(c) **[5 points]** Calculate the nonvanishing matrix elements of the perturbation between the ground state and the unbound states.

(d) **[5 points]** Using the Fermi golden rule, calculate the transition rate. Make sure the dimensionality of your final result is 1/time.

(e) **[5 points]** Sketch how the ionization rate depends on the frequency ω.

Potentially useful: $\int_0^\infty dx\, x \sin(ax) e^{-bx} = \frac{2ab}{(a^2+b^2)^2}$

Problem II.3

A particle of mass m is moving in a repulsive potential

$$V(r) = V_0 \frac{a^2}{r^2}, \quad V_0 > 0.$$

(a) [**3 points**] Write out the radial part of the Schrödinger equation.

(b) [**10 points**] The spatial dependence of the potential invites a variable substitution that transforms the answer from part (a) into an equation resembling the free-particle equation. Use such a substitution to find an exact expression for the partial wave phase shift, δ_ℓ.

(c) [**4 points**] Show that for $8mV_0a^2/\hbar \ll 1$ the phase shift can be approximated by

$$\delta_\ell = -\frac{\pi m V_0 a^2}{\hbar^2(2\ell+1)}.$$

(d) [**8 points**] In the same approximation, find an expression for the scattering amplitude in closed form.

Potentially useful:

- Asymptotic form of the spherical Bessel function: $\lim_{x\to\infty} j_\nu(x) = \frac{\sin(x - \nu\pi/2)}{x}$.

  Note that ν does not have to be an integer.

- The asymptotic form of $e^{i\mathbf{k}\cdot\mathbf{r}}$ is: $\sum_{\ell=0}^{\infty}(2\ell+1)i^\ell \frac{\sin(kr - \ell\pi/2)}{kr} P_\ell(\cos\theta)$.

  Here, θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{r}$.

- Also: $\sum_{\ell=0}^{\infty} P_\ell(\cos\theta) = \frac{1}{2\sin(\theta/2)}$.

Problem II.4

The "spin-orbit" interaction for a spin-1/2 particle is

$$H_{SO} = \frac{\hbar}{4m^2c^2}\nabla V \times \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}.$$

(a) [**2 points**] Recast this expression in terms of the vector components of ∇V, $\hat{\boldsymbol{p}}$, and $\boldsymbol{\sigma}$.

Now consider an electron bound to a central potential $V(r)$, in a state with orbital quantum number $\ell = 1$. In the following, it is convenient to use the real-valued orbital wavefunction basis $\{\psi_x, \psi_y, \psi_z\}$ (where the $\psi_{x,y,z}$ are linear combinations of $|\ell = 1, m = \pm 1, 0\rangle$ that transform like the x, y, z polar vector components).

(b) [**4 points**] Use spatial symmetry properties to find which term in your answer to (a) contributes to a nonzero matrix element $\langle \psi_y \uparrow | H_{SO} | \psi_z \downarrow \rangle = i\delta$, where δ is a constant common to all non-zero elements of H_{SO} in this basis (and depends on the details of $V(r)$). Why is $\langle \psi_y \uparrow | H_{SO} | \psi_z \uparrow \rangle = 0$, whereas $\langle \psi_y \uparrow | H_{SO} | \psi_x \uparrow \rangle \neq 0$?

(c) [**4 points**] Evaluate all the matrix elements of H_{SO} in the $\{\psi_x, \psi_y, \psi_z\}$ basis in terms of the common factor δ, and express H_{SO} as a 3×3 matrix of the appropriate 2×2 Pauli σ matrices.

(d) [**10 points**] Find the eigenvalues of H_{SO}, and show that they correspond to the $j = (\ell + s = 3/2), (\ell - s = 1/2)$ subspaces. Hint: Find the characteristic equation either by re-arranging the orbital $\otimes$ spin basis to express H_{SO} as $3 \oplus 3$ block diagonal, or employ the block determinant rules and Pauli commutation relations.

Now consider a perturbation whose angular dependence transforms as

$$x^2 + y^2 - 2z^2.$$

(e) [**5 points**] Describe how the $j = 3/2$ levels are split under the action of the perturbation. [No explicit calculation of matrix elements is needed, and this problem can be solved independently of the previous (a)-(d)].

Problem II.5

(a) **[5 points]** Using Newtonian gravity and classical mechanics, find the escape velocity from a spherically symmetric object of mass M and radius R. If the escape velocity is greater than the speed of light c, the object is a Newtonian "black hole". For a given mass M, express the maximum (or *Schwarzschild*) radius R_S in terms of M.

(b) **[5 points]** Hawking has predicted that a black hole is not really black, but radiates like a hot body at temperature T_H; the typical photon emitted has a wavelength close to the black hole radius. Estimate T_H in terms of $M, G, c, \hbar$ and k_B.

(c) **[10 points]** As a black hole radiates, it loses mass and shrinks in size, and the Hawking temperature goes up.

   (a) Assuming the black hole emits one photon of energy $k_B T_H$ in the amount of time it takes light to travel R_S, determine the lifetime of the black hole of initial mass M.

   (b) Suppose a black hole is created by a density fluctuation just after the big bang, $\approx 2 \times 10^{10}$ years ago. What must be its initial mass in order for it to be in the final stages of evaporation today?

(d) **[5 points]** The boundary between classical and quantum regimes (defining the *Planck* length ℓ_P) occurs when the radius approaches the black hole's Compton wavelength. Find ℓ_P in terms of the fundamental constants G, c, and $\hbar$, and estimate its value to within an order of magnitude in cm.

Possibly useful:

- Stefan-Boltzmann constant $\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$.
- Gravitational constant $G = 6.674 \times 10^{-11}$ N m^2/kg^2.