UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART II

August 27, 2010 **9 a.m. - 1 p.m.**

Do any four problems. Each problem is worth 25 points.

Put all answers on your answer sheets.

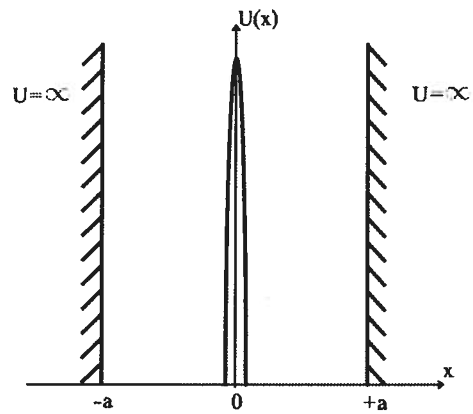
Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem II.1

Consider a quantum particle of mass m confined within a one-dimensional infinite square-well potential in the presence of an additional repulsive delta-potential in the middle of the well:

$$U(x) = \begin{cases} \beta\,\delta(x), & \text{if } -a < x < a; \\ \infty, & \text{if } |x| > a, \end{cases} \tag{1}$$

where $\delta(x)$ is the Dirac δ-function and $\beta \geq 0$. The potential $U(x)$ is illustrated below.



(a) **[2 points]** Consider the reflection operator $\hat{P}$, whose action on an arbitrary function $\psi(x)$ is defined as follows

$$\hat{P}\,\psi(x) = \psi(-x).$$

Determine the possible eigenvalues (called parity) of this operator.

(b) **[3 points]** Consider the stationary Schrödinger equation with the energy E corresponding to the potential (1). Show that the following wave functions satisfy the Schrödinger equation everywhere except, possibly, at the point $x = 0$

$$\psi_-(x) = \sin(k_- x), \tag{2}$$
$$\psi_+(x) = \sin\left[k_+(|x| - a)\right]. \tag{3}$$

Determine a relationship between the parameters $k_\pm$ and the energy E.

Show that the wave functions (2) and (3) have well-defined parities and determine the values of their parities.

(c) **[4 points]** For the potential (1), what boundary conditions does the wave function satisfy at $x = \pm a$?

Prove that the wave function satisfies the following matching condition at $x = 0$

$$\psi'(+0) - \psi'(-0) = \frac{2m\beta}{\hbar^2}\psi(0), \tag{4}$$

where the left-hand side represents the difference of the derivatives $\psi' = d\psi/dx$ taken in the limit $x \to 0$ from the positive side ($x > 0$) and from the negative side ($x < 0$).

II. 1 (Continued)

(d) **[3 point]** Apply the boundary conditions formulated above to the odd-parity wave function $\psi_-(x)$ from Eq. (2) and determine the permitted values of k_- and the corresponding eigenenergies E_-. Are the eigenenergies E_- affected by the presence ($\beta \neq 0$) or absence ($\beta = 0$) of the δ-function potential in the middle of the well?

(e) **[4 points]** Prove that the even-parity wave function $\psi_+(x)$ from Eq. (3) is an eigenfunction of the Schrödinger equation if the wave vector k_+ satisfies the following equation:

$$\tan(k_+a) = -\frac{k_+a}{\xi}, \quad \text{where} \quad \xi = \frac{ma\beta}{\hbar^2}. \tag{5}$$

From Eqs. (1) and (5), determine the dimensionality of β and ξ.

(f) **[3 points]** Examine Eq. (5) in the limit $\xi \to 0$, which corresponds to a vanishingly weak δ-function potential. Determine the permitted values of k_+ and the corresponding energy levels E_+ in this case. Compare your result with the well-known spectrum of an infinite potential well without δ-function potential ($\beta = 0$).

(g) **[3 points]** Now consider the limit of a very strong δ-function potential: $\xi \to \infty$. Determine the permitted values of k_+ and the corresponding energy levels E_+ from Eq. (5) in this limit. Compare with the spectrum of the odd-parity eigenstates found in Part (d).

(h) **[3 points]** Using the results obtained in Parts (d)–(g), make a qualitative plot showing how the energies of the four lowest states depend on the parameter ξ when it changes from 0 to ∞. Indicate the limiting values of the energies at $\xi = 0$ and $\xi = \infty$ on the plot. Give a qualitative description of the energy spectrum at $\xi \gg 1$.

Also indicate the parities of the four lowest states. What is the parity of the ground state?

Problem II.2

A rubidium atom has an excited state $|a\rangle$ of the same parity as the ground state $|g\rangle$, as shown in Figure 1. Because $|a\rangle$ and $|g\rangle$ have the same parity, the direct transition between these states is forbidden. However, the state $|a\rangle$ can decay to an opposite-parity state $|b\rangle$, which in turn can decay to the ground state $|g\rangle$. Suppose the states $|a\rangle$ and $|b\rangle$ are not populated initially. Then, at time $t = t_0$, the state $|a\rangle$ is instantly populated as $\rho_a(t_0)$. Suppose also that an experimental apparatus can detect emission of photons at the wavelength of the $|b\rangle \to |g\rangle$ transition, but is insensitive to the photons at the wavelength of the $|a\rangle \to |b\rangle$ transition.

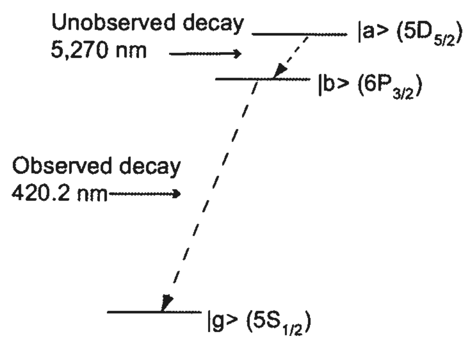


Figure 1: Energy levels and transitions involved in a cascade decay. The levels in parenthesis correspond to those relevant for the measurement in Rb.

(a) **[10 points]** Derive and solve equations for the time-dependent populations $\rho_a(t)$ and $\rho_b(t)$ of the states $|a\rangle$ and $|b\rangle$, given the lifetimes τ_a and τ_b of these states and the initial population $\rho_a(t_0)$. (The population $\rho_i(t)$ of a state i is the number of atoms in this state in an ensemble of atoms at the time t.)

From these solutions, determine the time dependence of the observed fluorescence signal, which is defined as the number of the $|b\rangle \to |g\rangle$ transitions per unit time.

(b) **[5 points]** Obtain the limiting forms of the expression for the time-dependent fluorescence signal for the cases: $\tau_a \gg \tau_b$, $\tau_a \ll \tau_b$, and $\tau_a \approx \tau_b$,

(c) **[10 points]** Figure 2 shows time dependence of the actual fluorescence signal measured at the wavelength $\lambda = 420$ nm in an experiment with atomic rubidium. The state $|a\rangle$ was populated by a short pulse that had a turn-off of less than 30 ns. The intermediate state $|b\rangle$ has the lifetime $\tau_b = 113$ ns. The decay $|a\rangle \to |b\rangle$ at $\lambda = 5,270$ nm is invisible to the detector, so the plot shows only the decay $|b\rangle \to |g\rangle$ at $\lambda = 420$ nm.

Use the graph to estimate the lifetime τ_a of the upper $|a\rangle$ state.

Hint: The lifetime τ_a of the $|a\rangle$ state is longer than the lifetime τ_b of the $|b\rangle$ state in Rb. Also, $\ln(10) = 2.3$.

II.2 (Continued)

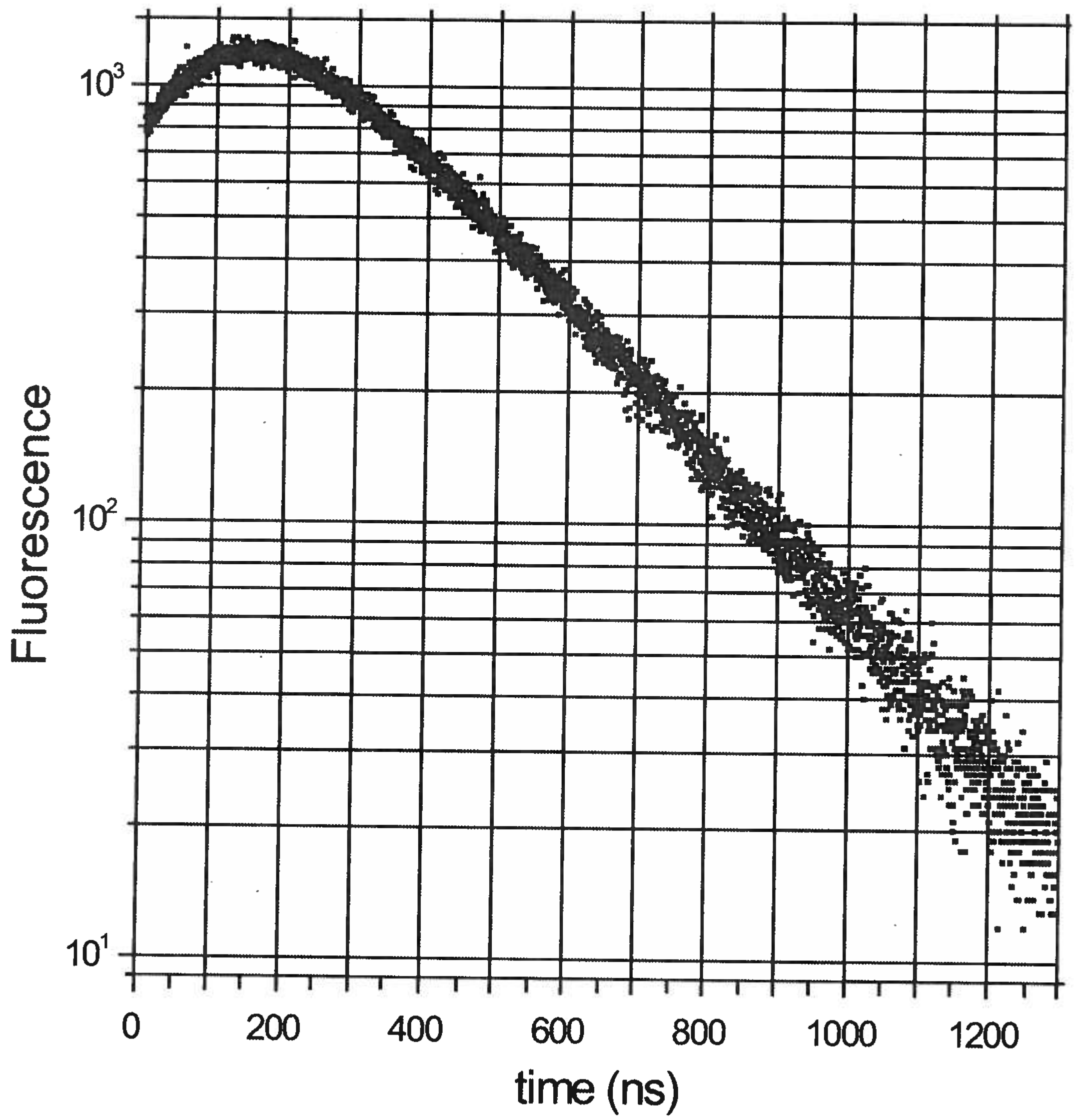


Figure 2: Observed fluorescence (the number of transitions per unit time) as a function of time for the $|b\rangle \rightarrow |g\rangle$ transition in Rb. Notice logarithmic scale on the vertical axis.

Problem II.3

This problem examines scattering in the high-energy limit, where the wavelength of the scattered particle is small, and quasiclassical approximation (or geometrical optics) is valid.

(a) **[5 points]** Consider a particle of mass m moving in the z direction. The incident wave function is $\psi(x, y, z) = e^{ikz}$, and the energy of the particle is $E = \hbar^2 k^2/2m$. The particle scatters on some three-dimensional potential $V(\boldsymbol{r})$.

When the particle's momentum $\hbar k$ is large, one can treat the particle's motion as one-dimensional along the z direction at the fixed coordinates x and y. In this approximation, consider the one-dimensional Schrödinger equation with the potential $V(x, y, z)$ and show that the wave function of the particle can be approximated by

$$\psi(x, y, z) = \exp\left(\frac{i}{\hbar}\int_{-\infty}^{z} dz' \sqrt{2m[E - V(x, y, z')]}\right). \tag{1}$$

(b) **[5 points]** Assuming that $E \gg V$, expand the integral in Eq. (1) to the first order in V and show that the wave function has the form

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \exp\left(-\frac{i}{v\hbar}\int_{-\infty}^{z} V(x, y, z')\, dz'\right), \tag{2}$$

where v is the particle's velocity, related to the energy by $E = mv^2/2$, and $\boldsymbol{k} = k\hat{z}$ is the particle's wave vector along the z axis.

The wave function (2) differs from the wave function of a free particle by the extra phase factor. This extra phase is called the eikonal phase, which depends on the transverse coordinate $\boldsymbol{b} = (x, y)$.

(c) **[5 points]** The scattering amplitude $f(\boldsymbol{k}, \boldsymbol{k}')$ from the state with the initial wave vector $\boldsymbol{k}$ to the state with the final wave vector $\boldsymbol{k}'$ can be calculated as

$$f(\boldsymbol{k}, \boldsymbol{k}') = -\frac{m}{2\pi\hbar^2}\int d^3r\, \psi_{\boldsymbol{k}}(\boldsymbol{r})\, V(\boldsymbol{r})\, e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}}. \tag{3}$$

(You do not need to derive Eq. (3).) By substituting Eq. (2) into Eq. (3) and taking the integral over z in Eq. (3), show that the scattering amplitude is

$$f(\boldsymbol{k}, \boldsymbol{k}') = \frac{k}{2\pi i}\int \left[\exp\left(-\frac{i}{v\hbar}\int_{-\infty}^{\infty} V(\boldsymbol{b}, z')\, dz'\right) - 1\right] e^{-i\boldsymbol{q}\cdot\boldsymbol{b}}\, d^2b. \tag{4}$$

Here $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$ is the momentum transfer, which is almost parallel to the $\boldsymbol{b}$ plane, so that $q_z \approx 0$. Eq. (4) gives the eikonal approximation for the scattering amplitude.

(d) **[5 points]** Let us now consider the following potential:

$$V(r) = \begin{cases} 0, & r > a \\ V_0, & r < a \end{cases} \tag{5}$$

where $V_0 > 0$, and $r = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$.

Using the eikonal approximation (4) and the optical theorem, calculate the total cross section σ for the potential (5). (You can leave the result as an integral over $\boldsymbol{b}$).

As a reminder, the optical theorem says that

$$\sigma = \frac{4\pi}{k} \mathfrak{Im} f(\boldsymbol{q} = 0), \tag{6}$$

where the right-hand side contains the imaginary part of the forward scattering amplitude.

(e) **[5 points]** In the very high-energy limit, where the phase in the first term of Eq. (4) is small, expand Eq. (4) to the first order in V and show that the eikonal approximation reduces to the Born approximation.

Formulate the condition of applicability of the Born approximation in the very high-energy limit for the potential (5) in terms of V_0, a, and v.

Problem II.4

This problem concerns properties of the states of quantum mechanical systems with a given, fixed value j of the quantum number of the angular momentum operator $\boldsymbol{J}$. Such a system will be referred to as a "spin-j system". (For example, the system could be a single particle, or an atom or molecule, etc. Other quantum numbers characterizing the system are not relevant for this problem.)

The questions first refer to what is implied, in terms of angular momentum, if such a system is in a spherically symmetric state. Then you are asked to consider states that have vanishing expectation value of the angular momentum vector. As you will see, these are distinct properties in general.

In your answers to this problem, all states should be normalized. Notation and basic facts about angular momentum are summarized under "possibly useful information" below. You may adopt units with $\hbar = 1$.

(a) **[5 points]** Among the systems of spin j, for general j, what systems admit spherically symmetric states? Display all such states in the basis $\{|j, m\rangle\}$.

(b) **[10 points]** Now consider a system composed of two (distinguishable) components, both with the same spin j. Find the spherically symmetric state of this system, expressed in two ways,

(i) using the eigenbasis of the composite system's total angular momentum, $|j_{\text{tot}}, m_{\text{tot}}\rangle$,

(ii) using products of the individual eigenstates, $|j, m_1\rangle|j, m_2\rangle$: Write the wave function of the system in the form

$$|\psi\rangle = \sum_{m_1, m_2} C_{m_1,m_2} |j, m_1\rangle|j, m_2\rangle$$

and determine the coefficients C_{m_1,m_2}.

(iii) Discuss the existence or non-existence of a spherically symmetric state if the two individual spins are unequal, $j_1 \neq j_2$.

(c) **[5 points]** Return to the case of a single spin-j system and consider expectation values. For a spin-1/2 system, find all the states (if any) with vanishing expectation value of all components of the vector $\langle \boldsymbol{J} \rangle$.

(d) **[5 points]** For contrast, consider a spin-1 system. Find at least one state with vanishing expectation value of the vector $\langle \boldsymbol{J} \rangle$, in the basis $|1, m\rangle$: Write the wave function of this state as $|\psi\rangle = \sum_m C_m |1, m\rangle$ and determine the coefficients C_m.
Is this state spherically symmetric? How can you obtain other such $\langle \boldsymbol{J} \rangle = 0$ states from this one?

Possibly useful information: The eigenstates $|j, m\rangle$ of $J^2 = \boldsymbol{J} \cdot \boldsymbol{J}$ and J_z for a spin-j system satisfy

$$J^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle, \qquad J_z|j, m\rangle = m\hbar|j, m\rangle$$

and

$$J_\pm|j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\, |j, m \pm 1\rangle$$

where $J_\pm = J_x \pm iJ_y$.

Problem II.5

Neutrinos can be observed in three different states $|\nu_e\rangle$, $|\nu_\mu\rangle$, and $|\nu_\tau\rangle$, which are called the flavor states. Recent experiments and theories indicate that these states are different from the energy eigenstates of neutrinos, which are denoted as $|\nu_1\rangle$, $|\nu_2\rangle$, and $|\nu_3\rangle$ and have different non-zero masses. For simplicity, we consider only the pairs $|\nu_\mu\rangle, |\nu_\tau\rangle$ and $|\nu_2\rangle, |\nu_3\rangle$ of the neutrino states. These pairs of states can be expressed as linear superpositions of each other, as follows

$$\begin{pmatrix} |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix}, \tag{1}$$

where the parameter θ is called the mixing angle. If neutrinos are initially in the state $|\nu_2\rangle$ or $|\nu_3\rangle$ at the time $t = 0$, then their wave function at a time t can be obtained as follows

$$\begin{aligned} |\nu_2\rangle &\to e^{-iE_2t/\hbar}\,|\nu_2\rangle, \\ |\nu_3\rangle &\to e^{-iE_3t/\hbar}\,|\nu_3\rangle, \end{aligned} \tag{2}$$

where E_2 and E_3 are the energies of the states $|\nu_2\rangle$ and $|\nu_3\rangle$, respectively. In a typical experiment, a beam of neutrinos is initially produced in the pure $|\nu_\mu\rangle$ state with momentum p. At a later time, neutrinos are detected in the state $|\nu_\tau\rangle$. This phenomenon is called flavor oscillation.

(a) **[5 points]** Using Eq. (1), explicitly write the states $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$ as superpositions of the states $|\nu_2\rangle$ and $|\nu_3\rangle$.

(b) **[5 points]** If a neutrino is created at $t = 0$ in the pure $|\nu_\mu\rangle$ state, obtain its wave function $|\psi(t)\rangle$ at a later time t in the basis of $|\nu_2\rangle$ and $|\nu_3\rangle$.

(c) **[10 points]** Suppose the state of the neutrino is measured at the time t. Using $|\psi(t)\rangle$ found in Part (b), calculate the probability amplitude and the probability of finding the neutrino in the state $|\nu_\tau\rangle$.

What is the maximal probability of finding the neutrino in the state $|\nu_\tau\rangle$ for any time t?

(d) **[5 points]** Using your answer to Part (c), find the minimum distance L required to maximize the probability of finding neutrinos in the state $|\nu_\tau\rangle$. Express your answer in terms of the difference in the masses m_2 and m_3 of the states $|\nu_2\rangle$ and $|\nu_3\rangle$. Assume that the momentum p is fixed and that the masses are very small, so that $m_2c, m_3c \ll p$ and the speed of neutrinos is very close to the speed of light c.

Possibly useful information:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$