UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART I

January 19, 2012 **9 a.m. - 1 p.m.**

Do any four problems.

Problems I. 1, I. 2, I. 4 and I. 5 are each worth 25 points.

Problem I. 3 Statistical Mechanics is worth 40 points.

Put all answers on your answer sheets.

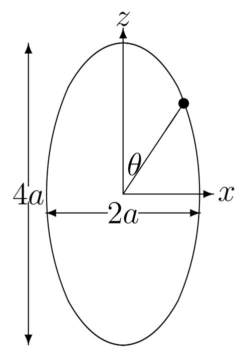
Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem I.1

Consider a frictionless bead of mass m that is constrained to move along an elliptical wire. The shape of the wire is given parametrically in terms of an angular variable θ as

$$x = a\sin(\theta)\,, \quad y = 0\,, \quad z = 2a\cos(\theta),$$

where a is a constant with dimensions of length. The wire sits in a gravitational potential of the form $V = mgz$; the maximum of the potential energy is $2mga$ at $\theta = 0$.



(a) [**6 points**] Write the Lagrangian for this system in terms of the variable θ and find the equations of motion.

(b) [**2 points**] Sketch the potential energy as a function of θ and write an expression for the total energy of the system.

For sufficiently high energies $E > E_{\text{th}}$, the system has no turning points, and the motion of the bead never stops. Find an expression for the threshold energy E_{th} in terms of the parameters of the model.

(c) [**5 points**] Linearize the equation of motion for small θ near $\theta = 0$ at the top of the wire. Show that the solution with the initial condition $\theta(0) = 0$ at time $t = 0$ is $\theta(t) = C\sinh(Dt)$, where C and D are constants. Find the constant D in terms of the parameters of the model. Find the constant C in terms of these parameters and the energy excess $\Delta E \equiv E - E_{\text{th}}$ of the particle.

(d) [**4 points**] For $E > E_{\text{th}}$, an exact expression for the period for one rotation as a function of the energy can be written in integral form: $\tau(E) = \int_0^{2\pi} f(\theta; E)\,\mathrm{d}\theta$. Use energy conservation to determine the function $f(\theta; E)$.

(e) [**8 points**] Suppose $E > E_{\text{th}}$, but $\Delta E \ll E_{\text{th}}$. Show that then

$$\tau \approx G\ln\left(\sqrt{\frac{\hat{E}}{\Delta E}}\right).$$

Express the constant G in terms of the parameters of the model and estimate $\hat{E}$ up to a dimensionless prefactor.

See next page for a hint

I.1 (Continued)

Hint: It is useful to note that for very small ΔE, the bead moves quite slowly at the top of the wire, *i.e.* near $\theta = 0$. Thus, the period of rotation is dominated by the time spent near the top. Let "near the top" mean $|\theta| < \theta_c$, where θ_c is a reasonably small (dimensionless) angle. If you simply neglect the time spent at $|\theta| > \theta_c$, you will still have a good approximation to the full period, even if the motion near the bottom is poorly described. Your approach may be based on Part (c) or on Part (d). The following formulas may be helpful:

$$\int_{-Y}^{Y} \frac{\mathrm{d}x}{\sqrt{A^2 + B^2x^2}} = \frac{2\sinh^{-1}\left(\frac{BY}{A}\right)}{B}.$$

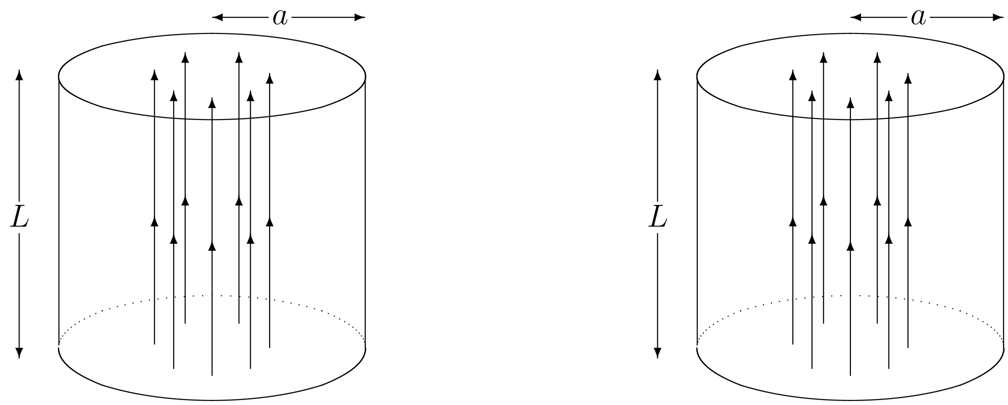
Asymptotic expressions at $x \to \infty :\quad \sinh(x) \to e^x/2, \quad \sinh^{-1}(x) \to \ln(2x).$

Problem I.2

We want to find cavity eigenmode frequencies for electromagnetic waves inside a closed cylindrical can of length L and radius a, consisting of side walls and top and bottom caps. The can is made from a highly conductive metal, whose electric conductivity is taken to be infinite in this problem: $\sigma = \infty$. $\{r, \varphi, z\}$ are cylindrical coordinates. Assume that the electric and magnetic fields in this problem are completely contained within the cavity, i.e. $\boldsymbol{E} = 0$ and $\boldsymbol{B} = 0$ outside the cavity.

Many of the parts of this problem can be done even if you skipped earlier parts. Useful equations are given at the end of the problem. The vacuum Maxwell Equations (ME) are:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \qquad c\,\boldsymbol{\nabla} \times \boldsymbol{B} = \partial_t\, \boldsymbol{E}, \qquad c\,\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t\, \boldsymbol{B}.$$



$\boldsymbol{E} = \hat{z} E_z(r)$ $\qquad$ $\boldsymbol{B} = \hat{z} B_z(r)$

(a) [**4 points**] At some time t, a snapshot is taken of the cavity, and an $\boldsymbol{E}$ field is observed to look as shown in the left figure, whereas $\boldsymbol{B} = 0$. The electric field is parallel to the cylindrical axis z of the cavity and depends only on r, but not on φ and z, i.e., $\boldsymbol{E} = \hat{z} E_z(r)$. If you were to run the time forward, what component of a $\boldsymbol{B}$ field would be generated? (Use your intuition and knowledge of the ME. No derivations.) Make a rough sketch of this emergent $\boldsymbol{B}$ field, including the direction.

(b) [**3 points**] State why the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields as obtained in Part (a) (given the symmetries) are consistent with the boundary conditions at the conducting walls. What is the value of E_z at $r = a$?

(c) [**6 points**] From one of the ME, find an equation for $\partial E_z / \partial t$ (under the symmetry assumptions). Check that this equation is consistent with your prediction of $\boldsymbol{B}$. Next, find the time evolution equation for your predicted component of $\boldsymbol{B}$. Combine the two equations to obtain the following wave equation for $E_z(r, t)$:

$$\frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right).$$

(d) [**4 points**] Assume solutions of the form $E_z(r, t) = \tilde{E}_z(r)\, e^{-i\omega t}$. Solve for the $\tilde{E}_z(r)$ eigenfunction using the boundary condition at $r = a$ and find the lowest eigenfrequency for this mode. This is the TM mode.

I.2 (Continued)

(e) [**4 points**] Now consider a putative configuration of a magnetic field $\boldsymbol{B}$ in the cavity as shown in the right figure (with $\boldsymbol{E} = 0$) for a supposed TE mode, under the same symmetry assumptions as in Part (a). What is wrong with this picture?

Suppose we still have $B_z \neq 0$, but we also allow for $\partial \boldsymbol{B}/\partial z \neq 0$, while preserving axial symmetry of the problem. Use one of the ME to determine which other component of $\boldsymbol{B}$ must be also nonzero (besides B_z). Write down this equation. Can you now sketch a more reasonable snapshot of the $\boldsymbol{B}$ field?

(f) [**4 points**] Run the time forward. What component of $\boldsymbol{E}$ field do you expect to be generated? Sketch this $\boldsymbol{E}$ field including the correct sign.

Useful equations: In cylindrical coordinates,

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}.$$

For an axially-symmetric field $\boldsymbol{F}(r, z)$, such that $\partial \boldsymbol{F}/\partial \varphi = 0$,

$$(\boldsymbol{\nabla} \times \boldsymbol{F})_r = -\frac{\partial F_\varphi}{\partial z}, \qquad (\boldsymbol{\nabla} \times \boldsymbol{F})_\varphi = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}, \qquad (\boldsymbol{\nabla} \times \boldsymbol{F})_z = \frac{1}{r}\frac{\partial (rF_\varphi)}{\partial r}.$$

A regular solution of the following equation

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

is given by the Bessel function of the zeroth order: $y(x) = J_0(x)$. The Bessel function vanishes $J_0(x_n) = 0$ at the points x_n, $n = 1, 2, \ldots$

Problem I.3

Consider a two-dimensional (2D) periodic crystal lattice consisting of a large number N of equivalent atoms and occupying an area A. The ratio $A/N \equiv a^2$ defines the characteristic length a of the order of interatomic distance. In this problem, we study contributions of lattice vibrations (phonons) to the thermal energy U and heat capacity C_V of the 2D crystal.

First consider the in-plane vibrations, where the atoms move in the 2D plane of the crystal. In the long-wavelength limit (for small k), the frequencies ω of these vibrational modes depend linearly on the 2D wavevector $\boldsymbol{k} = (k_x, k_y)$:

$$\omega_{\text{in}}(\boldsymbol{k}) = v\sqrt{k_x^2 + k_y^2} = vk, \tag{1}$$

where v is the speed of sound. There are two such modes (transverse and longitudinal), but we assume for simplicity that they are degenerate and have the same v.

(a) [**5 points**] In the Debye model, Eq. (1) is assumed to hold up to the Debye wavenumber k_D, i.e., to be valid for $k < k_D$. The value of k_D is determined by the requirement that the total number of vibrational modes in the circular domain $k < k_D$ is equal to the number $2N$ of the 2D spatial degrees of freedom of the atoms. Show that $k_D = 2\sqrt{\pi}/a$.

(b) [**8 points**] In the Debye theory, write an integral expression for the phonon energy $U(T)$, valid for all temperatures T. Also, write a general thermodynamic formula for the heat capacity at constant volume, $C_V(T)$, in terms of $U(T)$.

(c) [**8 points**] i) From your expressions in Part (b), find $U(T)$ and $C_V(T)$ in the low-temperature limit. ii) How does the T-dependence of $C_V(T)$ differ from the usual expression in three dimensions? iii) What is the relationship between the exponent of T in $U(T)$ and the spatial dimension? iv) What is the physical origin of this relationship?

(d) [**7 points**] From your expressions in Part (b), find $U(T)$ and $C_V(T)$ in the high-temperature limit and verify that they agree with the classical equipartition theorem.

(e) [**5 points**] Draw a sketch of $C_V(T)$ in the full range of temperatures, from low to high T, including $T = 0$. What is the characteristic temperature scale T_D (the Debye temperature) separating the low- and high-temperature limits?

The Nobel Prize in Physics in 2010 was awarded for the discovery of graphene, a 2D honeycomb lattice of carbon atoms. The figure on the next page shows the experimentally measured dispersion relations $\omega_n(\boldsymbol{k})$, $n = 1, \ldots, 6$, for the 6 vibrational eigenmodes in graphene. The modes represented by Eq. (1), with different values of v, correspond to the second and third lowest curves near the origin. (The upper three branches are due to the two-atom unit cell in a honeycomb lattice. Ignore these three upper branches, because they are not excited at low temperatures.) The lowest branch originates from the out-of-plane motion of the atoms perpendicular to the 2D plane. Similarly to perpendicular vibrations of an elastic plate, this mode has the following dispersion relation for small k:

$$\omega_{\text{out}}(\boldsymbol{k}) = b\,k^2, \tag{2}$$

where b is a coefficient.

I.3 (Continued)

(f) [**7 points**]. Determine temperature dependences of the contributions from the mode in Eq. (2) to $U(T)$ and $C_V(T)$ at low T. Sketch the contribution to $C_V(T)$ by a dashed line on your plot in Part (e) for low T only. Which mode gives the predominant contribution to $C_V(T)$ at low T, the in-plane mode (1) or the out-of-plane mode (2)?

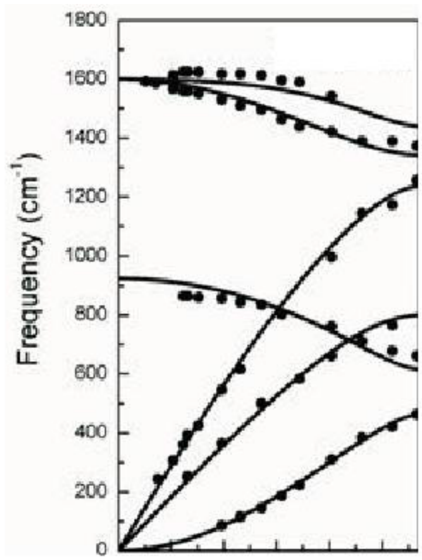


Figure 1: Phonon dispersion relations $\omega_n(\boldsymbol{k})$, $n = 1, \ldots, 6$, in graphene.

Problem I.4

In this problem, you are to consider the decay of one particle into two particles, one of which has zero mass, using the special theory of relativity and an alternative theory.

(a) **[10 points]** Einstein's special theory of relativity does not allow the decay of a particle A of mass m_A to two particles B and C (of masses m_B and m_C respectively) if $m_C = 0$ and $m_A < m_B$. Prove this statement using energy and momentum conservation in two ways: in the rest frame of the particle A and also in a frame where the energies of the particles $E_{A,B}$ are much larger than their masses.

Recently, it has been claimed by the OPERA experiment in Italy that neutrinos travel faster than the speed of light. One way to possibly understand this new result is to replace the energy-momentum relation of special relativity by the following equation in the laboratory reference frame:

$$E_A^2 = m_A^2 c^4 + p_A^2 c^2 (1 + \epsilon_A), \tag{1}$$

where $\epsilon_A \ll 1$. The ϵ-factors for different species could be different. In what follows, assume that $\epsilon_{B,C} = 0$ whereas $\epsilon_A \neq 0$. Also assume that $m_C = 0$ and $m_A < m_B$, as in Part (a). Note that Eq. (1) violates Lorentz invariance, so answer the following questions in the laboratory reference frame.

(b) **[6 points]** The derivative relation between energy, momentum, and velocity in classical physics, which follows from identifying the energy as the Hamiltonian and using the Hamilton equation $\boldsymbol{v} = dH/d\boldsymbol{p}$, gives the particle speed v. Using Eq. (1), find an expression for the speed v_A of particle A and show that its maximum value can exceed the speed of light c. Find the maximum speed of particle A in terms of ϵ_A.

(c) **[9 points]** Show that, if nature obeyed the above form of violation of the special theory of relativity, then particle A, traveling at a high enough speed, can decay to particles B and C (which is kinematically forbidden in the $\epsilon_A = 0$ limit). Calculate the threshold momentum p_A^{th} and energy E_A^{th} of particle A for this decay to take place, in terms of m_A, m_B, c, and ϵ_A. Assume energy and momentum conservation and work in the laboratory reference frame, where Eq. (1) holds.

To answer this question, plot the energy-momentum relations E vs. p for both particles A and B on the same graph from $p = 0$ to a sufficiently high p. Do these two curves intersect? If so, denote the momentum at the intersection point as p_0.

Is it permitted, by the energy-momentum conservation laws, for particle A to decay to particles B and C if $p_A = p_0$? What are the momentum and energy of particle C in this case? Is the decay process permitted for $p_A < p_0$? For $p_A > p_0$?

Given your answers to these questions, conclude that p_0 gives the threshold momentum: $p_A^{\text{th}} = p_0$. Calculate p_0 and the corresponding energy E_A^{th} in terms of the parameters of the problem.

Problem I.5

In this problem, we study properties of electromagnetic waves. Maxwell's equations for the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in vacuum in Gaussian units are

$$\frac{\partial \boldsymbol{E}}{\partial t} = c\,\boldsymbol{\nabla} \times \boldsymbol{B}, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = -c\,\boldsymbol{\nabla} \times \boldsymbol{E}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0. \tag{1}$$

(a) (a) [**5 points**] Show that in vacuum both $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy wave equations. What is the wave velocity?

(b) [**4 points**] Write a plane-wave solution for a given wavevector $\boldsymbol{k}$. What are the possible directions of $\boldsymbol{E}$ and $\boldsymbol{B}$ relative to $\boldsymbol{k}$ and to each other? How many linearly independent directions are there for, say, $\boldsymbol{E}$?

(b) [**5 points**] The energy density in an electromagnetic field is $u = (E^2 + B^2)/8\pi$ and the energy flux, or Poynting vector, is $\boldsymbol{S} = c\,(\boldsymbol{E} \times \boldsymbol{B})/4\pi$. Using Maxwell's equations, show that u and $\boldsymbol{S}$ satisfy the continuity equation for energy.

(c) Here we consider electromagnetic radiation caused by an oscillating electric dipole of magnitude p located at the origin and aligned along the z axis. Assume that the dipole (and hence the fields) vary sinusoidally in time $\sim e^{-i\omega t}$. In the approximation where the scale of the dipole, d, is small, $kd \ll 1$, and in the radiation zone, $kr \gg 1$, the vector potential in a particular gauge at a distance r from the dipole is

$$\boldsymbol{A}(\boldsymbol{r}, t) = -ikp\,\frac{e^{ikr - i\omega t}}{r}\,\hat{\boldsymbol{z}}. \tag{2}$$

Here $k = \omega/c$ is the wavenumber. There is also a scalar potential that is not explicitly given here.

(a) [**4 points**] In the radiation zone, $kr \gg 1$, what are the electric and magnetic fields? Obtain $\boldsymbol{B}$ using $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, and $\boldsymbol{E}$ from Eq. (1).

(b) [**3 points**] What is the energy flux $\boldsymbol{S}$ of the waves for $kr \gg 1$?

(c) [**4 points**] What is the time-averaged power $dP/d\Omega$ radiated per unit solid angle by the oscillating dipole moment for $kr \gg 1$?

Some useful vector calculus identities for arbitrary $\boldsymbol{W}$ and $\boldsymbol{V}$ are

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{W}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{W}) - \boldsymbol{\nabla}^2 \boldsymbol{W}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{W}) = 0$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{W} \times \boldsymbol{V}) = \boldsymbol{V} \cdot (\boldsymbol{\nabla} \times \boldsymbol{W}) - \boldsymbol{W} \cdot (\boldsymbol{\nabla} \times \boldsymbol{V})$$

In spherical coordinates,

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$$

in terms of spherical unit vectors, and the curl is

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \frac{\hat{\boldsymbol{r}}}{r \sin\theta}\left[\frac{\partial(V_\phi \sin\theta)}{\partial\theta} - \frac{\partial V_\theta}{\partial\phi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial(rV_\phi)}{\partial r}\right] + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial(rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right].$$