UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART II

January 18, 2013 **9 a.m. - 1 p.m.**

Do any four problems. Each problem is worth 25 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem II.1

A deuteron consists of a proton and a neutron forming a weakly bound system with no excited bound states. The nuclear interaction between the proton and neutron can be approximated by a spherically-symmetrical square-well potential $V(r)$:

$$V(r) = \begin{cases} -V_0, & r \leq a \\ 0, & r > a \end{cases} \tag{1}$$

The proton and neutron can be treated as non-relativistic particles in this problem.

(a) **[5 points]** Write down the radial Schrödinger equation for the radial wavefunction $R(r)$ describing the relative motion of the proton and neutron. Then make the substitution $u(r) = rR(r)$ and write down the Schrödinger equation for $u(r)$ in the $\ell = 0$ state.

(b) **[12 points]** Applying the appropriate boundary conditions, find expressions for the wavefunction $u(r)$ of a bound state and the condition that determines the energies of bound states for the potential given by Eq. (1).

(c) **[6 points]** Determine the minimum value of V_0 for which just one bound state exists with a very small binding energy $|E| \ll V_0$.

(d) **[2 points]** The range of the nuclear force is estimated to be $a \approx 2$ fermi in Eq. (1). Assuming the weak binding condition $|E| \ll V_0$, numerically estimate the potential depth V_0 in MeV for the deuteron.

Useful constants:
1 fermi $= 10^{-15}$ m,:
$\hbar = 197$ MeV fermi/c $= 1.0546 \times 10^{-37}$ Js,
Mass of Proton: 938.3 MeV$/c^2 = 1.6726 \times 10^{-27}$ kg,
Mass of Neutron: 939.6 MeV$/c^2 = 1.6749 \times 10^{-27}$ kg.

Problem II.2

For a one-dimensional harmonic oscillator of mass m and frequency ω, the two lowest energy eigenstates have the energies $\hbar\omega/2$ and $3\hbar\omega/2$, and their wavefunctions are

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \text{and} \qquad \psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}.$$

(a) **[5 points]** Consider the harmonic oscillator potential $V_0 = \frac{1}{2}m\omega^2(x^2+y^2)$ in two dimensions. Write down the Hamiltonian H_0 and find the energies and wavefunctions of the ground and first excited states. What are the degeneracies of these states?

(b) **[8 points]** Suppose a weak perturbation $V_1 = \lambda xy$ is introduced, so that the potential changes to $V = V_0 + V_1$. Using a perturbation theory (degenerate if appropriate), compute the first-order change in the energies for the states found in Part (a).

(c) **[7 points]** Obtain the exact energy eigenvalues for the total Hamiltonian with the full potential $V = V_0 + V_1$. Verify that your exact result agrees with the result in Part (b) to the first order in λ.

Hint: Consider the substitutions

$$x' = \frac{x+y}{\sqrt{2}}, \qquad y' = \frac{x-y}{\sqrt{2}}, \qquad \text{and} \qquad p'_x = \frac{p_x+p_y}{\sqrt{2}}, \qquad p'_y = \frac{p_x-p_y}{\sqrt{2}}.$$

(d) **[5 points]** Compare the Hamiltonians in Parts (a) and (c). Comment in a few sentences on the symmetry and degeneracy of the systems they describe.

$$\text{Gaussian integrals}: \qquad \int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} s^2 e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

Problem II.3

Consider s-wave scattering of a particle of mass m on the delta-function spherical-shell potential of radius a and strength $U > 0$,

$$V(r) = U\,\delta(r-a). \tag{1}$$

(a) [**7 points**] Write down the $\ell = 0$ radial Schrödinger equation for the function $u(r) = r\psi(r)$ and an energy $E = \hbar^2 k^2/2m$.

Formulate the boundary conditions on u and du/dr at $r = 0$ and at $r = a$, using the abbreviation $\gamma = (2m/\hbar^2)\,U$.

Show that $u(r)$ has the following form, where δ_0 is called the scattering phase:

$$u(r) = \begin{cases} A\sin(kr), & r \le a, \\ B\sin(kr+\delta_0), & r \ge a. \end{cases} \tag{2}$$

Applying the boundary conditions to the wavefunction (2), derive a transcendental equation for δ_0.

The scattering phase δ_0 is related to the cross-section of scattering σ by

$$\sigma = \frac{4\pi}{k^2}\,\sin^2\delta_0. \tag{3}$$

In the rest of the problem, assume a very strong scattering potential, so that $\gamma \gg k,\ 1/a$.

(b) [**6 points**] From your results in Part (a), obtain the phase shift δ_0 in the case where $\tan(ka)$ is not close to zero. Show that δ_0 in this case is the same as for an impenetrable, hard-sphere scattering potential of radius a.

Find the cross-section of scattering σ in the low-energy limit $ka \ll 1$. Compare your result with scattering cross-section of classical particles on a hard sphere of radius a.

(c) [**6 points**] When $\tan(ka)$ is close to zero, show that there are resonances in scattering for certain values of k. Calculate δ_0 and σ for a resonance and show that σ is maximal with respect to k.

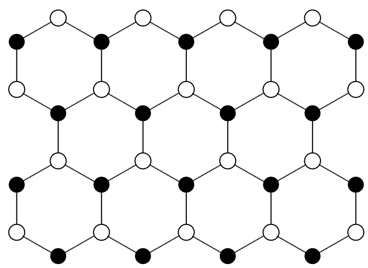
(d) [**6 points**] Calculate the energies E_n of the bound states with $\ell = 0$ inside the spherical-shell potential in Eq. (1) with an infinite impenetrable wall $U = \infty$. Compare these energies E_n with the energies of resonant scattering in Part (c). Explain the connection.

Problem II.4

The Nobel Prize in Physics 2010 was awarded for the discovery of graphene, a single-layer hexagonal lattice of carbon atoms shown in the figure below. It has two sub-lattices A and B shown by open and closed circles, so the electron wavefunctions are described by two-component spinors $\psi = (\psi_A, \psi_B)$. Near the Fermi energy, the Hamiltonian for the electrons in graphene can be approximated as

$$\mathcal{H} = v\,(\boldsymbol{\sigma} \cdot \boldsymbol{p}),$$

where v is the Fermi velocity, $\boldsymbol{p} = (p_x, p_y) = -i\hbar\boldsymbol{\nabla}$ is the two-dimensional momentum operator, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ are the Pauli matrices operating on the spinor wavefunctions.



(a) [**5 points**] Find the eigenenergies E and spinor eigenfunctions $\psi(x, y)$ of the Schrödinger equation for the electron in graphene: $\mathcal{H}\psi = E\psi$. Show that the eigenenergies $E(\boldsymbol{p})$ are linear in the magnitude of the electron momentum p.

How does the relative phase between the spinor components ψ_A and ψ_B depend on the direction of $\boldsymbol{p}$?

(b) [**5 points**] Suppose the electron encounters a (sub-lattice independent) arbitrary potential $V(x)\,\sigma_0$, where σ_0 is the 2×2 unit matrix, with the asymptotic limits

$$V(x) = \begin{cases} 0 & \text{at } x \to -\infty, \\ V_0 & \text{at } x \to +\infty, \end{cases} \qquad V_0 > 0.$$

Assuming $p_y = 0$, write down (but do not solve) the coupled Schrödinger equations for the spinor components of $\psi(x)$ in the presence of $V(x)$.

(c) [**5 points**] Now recognize that a unitary transformation of the spinor wavefunction can be performed to transform the Pauli matrix σ_x into σ_z, thus diagonalizing the Hamiltonian. Such a transformation can be accomplished with the operator

$$R = e^{i(\boldsymbol{\sigma}\cdot\boldsymbol{n})\chi/2} = \sigma_0 \cos\left(\frac{\chi}{2}\right) + i(\boldsymbol{n} \cdot \boldsymbol{\sigma}) \sin\left(\frac{\chi}{2}\right),$$

where $\boldsymbol{n}$ a unit vector for the axis of rotation ($n = 1$), and χ is a rotation angle.

What choice of $\boldsymbol{n}$ and χ achieves the desired transformation? Write down the transformed Hamiltonian and Schrödinger equation, but do not solve.

II.4 (Continued)

(d) [**5 points**] Finally, the wavefunction can be modified by a phase factor and written as

$$\psi(x) = e^{-i(\sigma_z/v\hbar)\int^x V(x')dx'}\,\tilde{\psi}(x).$$

Obtain the Schrödinger equation for the modified spinor wavefunction $\tilde{\psi}(x)$ and show that it does not contain $V(x)$, i.e. corresponds to a free particle.

(e) [**5 points**] Using the result from Part (d), state the reflection and transmission coefficients of the potential $V(x)$. Provide a physical explanation of the result in the case $0 < E < V_0$.

This effect is known as the Klein paradox.

Useful information:

Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Problem II.5

The neutrinos produced in weak interactions are flavor eigenstates denoted as $|\nu_\mu\rangle$ and $|\nu_e\rangle$. An experiment produces isotropically-distributed pure $|\nu_\mu\rangle$ (muon neutrinos). In a year, a small-size detector observes 40,000 muon neutrinos at a distance of 100 meters from the source and 300 muon neutrinos at 1000 meters from the source.

(a) **[3 points]** Based on the data for 100 meters, what number of muon neutrinos would you expect at 1000 meters, if the muon neutrinos are conserved and emitted isotropically?

Could the difference between the expected and observed numbers of neutrinos be due to statistical fluctuations in the number of observations? Discuss, in terms of standard deviation, the strength of the evidence that muon neutrinos "disappear" (i.e. are not conserved) between 100 and 1000 meters.

Neutrinos of one flavor can disappear because flavor eigenstates can oscillate between each other as they propagate through space. A flavor eigenstate does not have a well-defined mass; it is a superposition of eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ that have definite masses m_1 and m_2. In the two-dimensional Hilbert space of mass eigenstates, the flavor eigenstates make a constant angle θ (called the mixing angle) with the mass eigenstates, as shown in Fig. 1:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}.$$

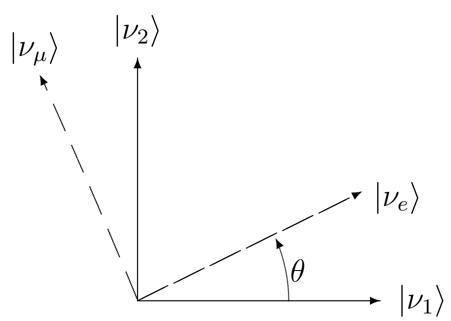


Figure 1: Neutrino eigenstates represented as rotation between the flavor and mass eigenstates.

Here we study neutrinos with a well-defined momentum p, which is conserved. The two mass eigenstates have energies $E_1(p)$ and $E_2(p)$, whereas the two flavor eigenstates of momentum p do not have well-defined energies.

(b) **[8 points]** Show that the state that is a pure muon neutrino at $t = 0$, $|\psi(0)\rangle = |\nu_\mu\rangle$ evolves according to

$$|\psi(t)\rangle \;=\; -\sin\theta|\nu_1\rangle e^{-iE_1 t} + \cos\theta|\nu_2\rangle e^{-iE_2 t}$$

Hence calculate the probability $P(\nu_\mu \to \nu_e)$ of a transition from $|\nu_\mu\rangle$ to $|\nu_e\rangle$ as a function of time t, in terms of θ and $E_{1,2}$.

(c) [**8 points**] Assuming that $m_1,\, m_2 \ll p/c$, find the lowest-order contribution to the energies $E_{1,2}$ due to non-zero masses of neutrinos. Substitute the result into the formula for $P(\nu_\mu \to \nu_e)$ obtained in Part (b). Show that $P(\nu_\mu \to \nu_e)$ oscillates as function of the neutrino's travel distance $L = ct$ according to

$$P(\nu_\mu \to \nu_e) \;=\; \sin^2 2\theta\, \sin^2\left(\frac{\Delta m^2 L}{4E_\nu}\right),$$

where $\Delta m^2 = m_1^2 - m_2^2$, and E_ν is an average neutrino energy.
Helpful Hint: $(1 - \cos\, 2\theta)/2 = \sin^2\theta$

(d) [**4 points**] Consider the data from a neutrino oscillation experiment shown in Fig. 2, where the *survival probability*, $1 - P(\nu_\mu \to \nu_e)$, is plotted vs L/E_ν. Assuming the error bars represent 1-sigma errors, roughly estimate by eye the χ^2 as well as the number of degrees of freedom between the data and the two hypotheses: i) the best-fit oscillation hypothesis shown as the curve for two fit parameters, ii) the best-fit flat distribution hypothesis shown as the horizontal dashed line. Discuss what this implies for the two hypotheses.

(e) [**2 points**] For the better hypothesis estimate $\Delta m^2 c^4$ in eV^2. (Notice that the horizontal axis does not start from zero.)

Useful data: $hc = 1.24 \times 10^{-6}$ eV m.

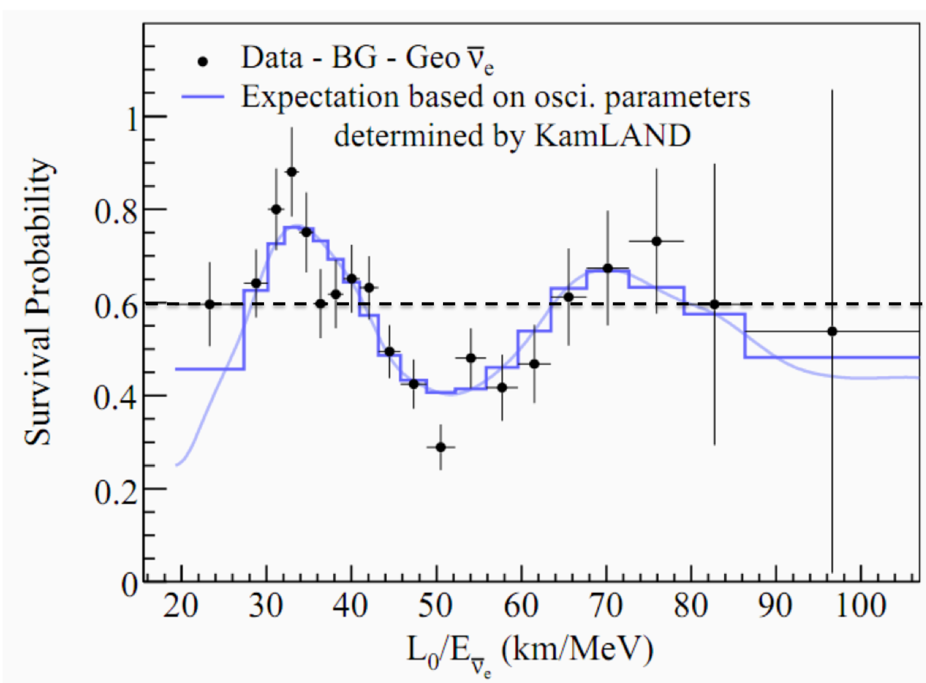


Figure 2: Data from a neutrino oscillation experiment.