UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

August 29th, 2014 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

Problem II.1

The Schrödinger equation for the helium atom cannot be solved exactly. However, if we replace each of the Coulomb forces by a spring force, the system *can* be solved exactly. As an example, consider the Hamiltonian H in 3-dimensional space given by

$$H = -\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right) + \frac{1}{2}m\omega^2\left(r_1^2 + r_2^2\right) - \frac{\lambda}{4}m\omega^2|\mathbf{r}_1 - \mathbf{r}_2|^2 \ , \tag{1}$$

with $0 < \lambda < 1$. Here $\mathbf{r}_1$ and $\mathbf{r}_2$ represent the coordinates associated with the two electrons and m is the electron mass.

(a) [**5 points**] Neglecting the last term (i.e. setting $\lambda = 0$) in H, determine the ground state energy of the two (no longer coupled) 3D harmonic oscillators. Also write down the ground state wave function.

(b) [**5 points**] Use this uncoupled ground state wave function and first order perturbation theory to estimate the ground state energy of the full, coupled Hamiltonian H.

(c) [**10 points**] By a suitable change of variables, show how the full Hamiltonian H can be transformed into the sum of two independent simple harmonic oscillators in 3D,

$$H = \left[-\frac{\hbar^2}{2m}\nabla_u^2 + \frac{1}{2}m\omega^2u^2\right] + \left[-\frac{\hbar^2}{2m}\nabla_v^2 + \frac{1}{2}\left(1-\lambda\right)m\omega^2v^2\right] \ . \tag{2}$$

Explicitly give the $u = u(\mathbf{r}_1, \mathbf{r}_2)$ and $v = v(\mathbf{r}_1, \mathbf{r}_2)$ that satisfy this transformation.

(d) [**5 points**] Determine the exact ground state energy of the system. How well does the answer agree with your estimate from part (b) above?

Note: The normalized ground state wave function of a single harmonic oscillator in one dimension is given by

$$\psi(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \exp\left(-\alpha x^2/2\right) \ , \tag{3}$$

where $\alpha = m\omega/\hbar$.

Problem II.2

Linearly polarized photons with energy $\hbar\omega \gg \epsilon_B$ are incident on a target of deuterium, which has a proton-neutron binding energy of ϵ_B. The radiation affects the quantum state of the proton. (Ignore the electron in this problem.) Here you will apply time dependent perturbation theory to the perturbation

$$H' = -\frac{e}{Mc}\mathbf{p}\cdot\mathbf{A}, \qquad \mathbf{A} = A_0\hat{\mathbf{x}}\cos(kz-\omega t), \tag{1}$$

where e, M, and $\mathbf{p}$ are the proton's charge, mass and momentum, respectively. c is the speed of light, and $\mathbf{A}$ is the vector potential of the incident radiation. Gaussian units are used in this problem.

(a) [**4 points**] The energy density in an electromagnetic wave is $u = E^2/4\pi$, with $\mathbf{E}$ the electric field. Using this, show that the time-averaged number flux of incident photons is

$$\frac{\omega}{8\pi\hbar c}A_0^2. \tag{2}$$

(b) [**10 points**] Neglecting all spin effects and using Fermi's golden rule, obtain a general expression for the differential cross section for the emission of a proton into a plane wave continuum state, $\phi_{\mathbf{k}_f}$, with momentum $\mathbf{k}_f$ in terms of the (not yet specified) deuteron ground state wave function ψ_0.
Hint: Energy conservation implies that only the term containing $e^{-i\omega t}$ in the vector potential is relevant for this problem.
Hint: Because the transition is to a plane wave continuum state, you will need to calculate a density of states—a convenient approach is to assume box quantization.

(c) [**3 points**] What is the general angular dependence of the differential cross section?

(d) [**5 points**] From (b) obtain an explicit formula for the photon-proton total cross section as a function of $\hbar\omega$ using for the deuteron ground state wave function

$$\psi_0(r) = \sqrt{\frac{\kappa}{2\pi}}\frac{1}{r}e^{-\kappa r}, \tag{3}$$

which corresponds to an attractive delta-function potential between the neutron and proton. Here $\hbar^2\kappa^2/M = \epsilon_B$ and r is the neutron-proton separation.

(e) [**3 points**] Given the potential described in part (d), explain why the plane wave approximation is accurate for the matrix element of part (b).

Problem II.3

In his famous experiment of 1910, E. Rutherford scattered 5 MeV α particles from gold foil. He later wrote: "It was quite the most incredible event that happened to me in my life. It was as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you".

(a) **[3 points]** Given the era in which the experiment was performed, explain clearly and carefully why Rutherford was so surprised.

(b) **[12 points]** Due to the presence of valence electrons in the gold atoms, the positive nuclear charge is compensated at large distances. This can be modeled with a "screened" Coulomb potential:

$$V(r) = V_0 \frac{e^{-Ar}}{r}, \quad A \geq 0.$$

Derive the differential scattering cross section (up to a factor of order unity but with correct units) from this potential in the first Born approximation.

(c) **[7 points]** Using your result from (b), calculate the total cross section. What happens in the limit of zero screening ($A = 0$), and why?

(d) **[1 point]** Why is it important that this backscattering experiment is performed on a *thin* gold foil?

(e) **[2 points]** Suppose we perform the Rutherford experiment using an ordered crystal of gold (instead of a thin amorphous film) and a highly collimated beam of α-particles directed along a high symmetry axis. By considering the de Broglie wavelength of the 5 MeV α particles, describe what you would expect to observe in such an experiment.

Problem II.4

This problem studies degeneracy of the energy levels of a spherically-symmetrical harmonic oscillator in three dimensions. The eigenenergies E can be easily obtained in cartesian coordinates as $E = (N + \frac{3}{2})\hbar\omega$, $N = N_x + N_y + N_z$, where N_x, N_y, and N_z are the integer quantum numbers for each cartesian direction. Clearly, there is a high degree of degeneracy for the excited energy levels.

(a) **[5 points]** Because the problem is spherically symmetrical, the eigenstates $|L, M\rangle$ of the orbital angular momentum are also the energy eigenstates. What values of L are present in a degenerate multiplet specified by N? Consider even and odd N.

(b) **[5 points]** Calculate degeneracy of the $N = 2$ multiplet, using

(i) the cartesian quantum numbers N_x, N_y, and N_z;

(ii) the spherical quantum numbers L and M.

Verify that both methods produce the same result.

(c) **[5 points]** Now suppose that a small perturbation breaks spherical symmetry of the system. Consider the following Hamiltonian of the perturbation

$$H_1 = \alpha\, r^5\, Y_5^0(\theta, \phi), \tag{1}$$

where $Y_l^m(\theta, \phi)$ is a spherical harmonic, and α is a small parameter.

To the first order in α, does the perturbation H_1 split energy levels of the degenerate multiplets specified by N?

You do not need to evaluate any integrals, but should explain your results on the basis of symmetry considerations here and below.

(d) **[5 points]** Now, instead of (1), suppose that a Hamiltonian of perturbation is

$$H_2 = \beta\, r^6\, Y_6^0(\theta, \phi). \tag{2}$$

Is M a good quantum number in the presence of the perturbation H_2?

Is there a relation between eigenenergies of the states with M and $-M$?

(e) **[5 points]** For $N = 3$, how do the following states split to the first order in β due to the perturbation H_2?

(i) the states $|L, M\rangle$ with $L = 1$,

(ii) the states $|L, M\rangle$ with $L = 3$.

Problem II.5

Consider a set of N neutral fermions (for example, neutrons) of mass m in a cubical box of side L. (You may use fixed or periodic boundary conditions; state clearly which you choose.) Assume that the fermions are nonrelativistic and non-interacting.

(a) [**4 points**] Write down expressions for
(i) the allowed wave vectors of the fermions.
(ii) their allowed energy levels.

(b) [**5 points**] Using the answers to (a) and the Pauli principle, obtain a relationship between the maximum occupied energy level, the so-called Fermi energy ϵ_F (or the corresponding Fermi wave number k_F), and the number density of the fermions $n \equiv N/V$ at zero temperature, i.e. $T = 0$.

(c) [**5 points**] From (b), argue qualitatively that the fermions show an outward pressure at $T = 0$, the so-called degeneracy pressure, which resists compression. Then calculate explicitly the total energy $\mathcal{E}(N, V)$ and thereby the degeneracy pressure for N non-relativistic fermions in a box of volume V in terms of N and V. [Note that the shape of the box does not matter for large V.]

(d) [**6 points**]
(i) Using Newtonian physics, calculate the (inward) gravitational "pressure" of a (spherical) star composed of such fermions that has a mass M and volume $V = (4/3)\pi R^3$, with uniform density $\rho = M/V$ and radius R. Express your answer in extensive variables (not ρ).
(ii) From the condition of balance between the gravitational pressure and the degeneracy pressure, find the relationship between N and V for the [neutron] star and thence the dependence of R on M. (The relation between the two should involve only numbers and fundamental constants, not other variables.)

(e) [**2 points**] For the assumption of nonrelativistic behavior to be valid what inequality should $\mathcal{E}/N$ satisfy?

(f) [**2 points**] Suppose the fermion has spin 3/2 (a hypothetical assumption since none has been discovered yet). How would your answer for the degeneracy and the gravitational pressure change? (Just a short answer. Indicate how the key parameters change.)

(g) [**1 point**] Suppose the particle has spin 1. How would your answer change? (Short answer. Do not redo the whole problem.)